Curvature

Riemann curvature tensor

Measures the difference between the flat spacetime and the given one in a coordinate-invariant way

Distinguishes flat spacetimes in non-Cartesian coordinates from other spacetimes

Tensorial object, may be defined in many ways

Plays a crucial role in GR and differential geometry

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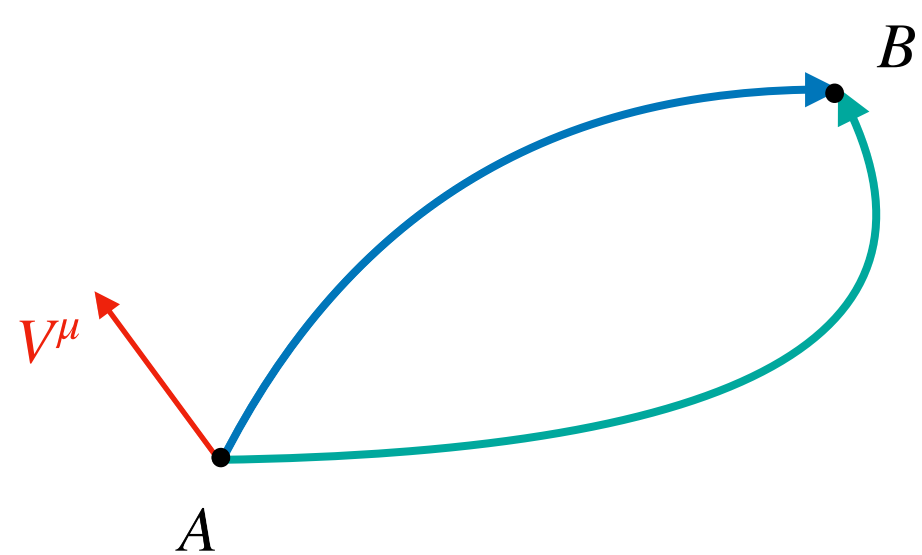
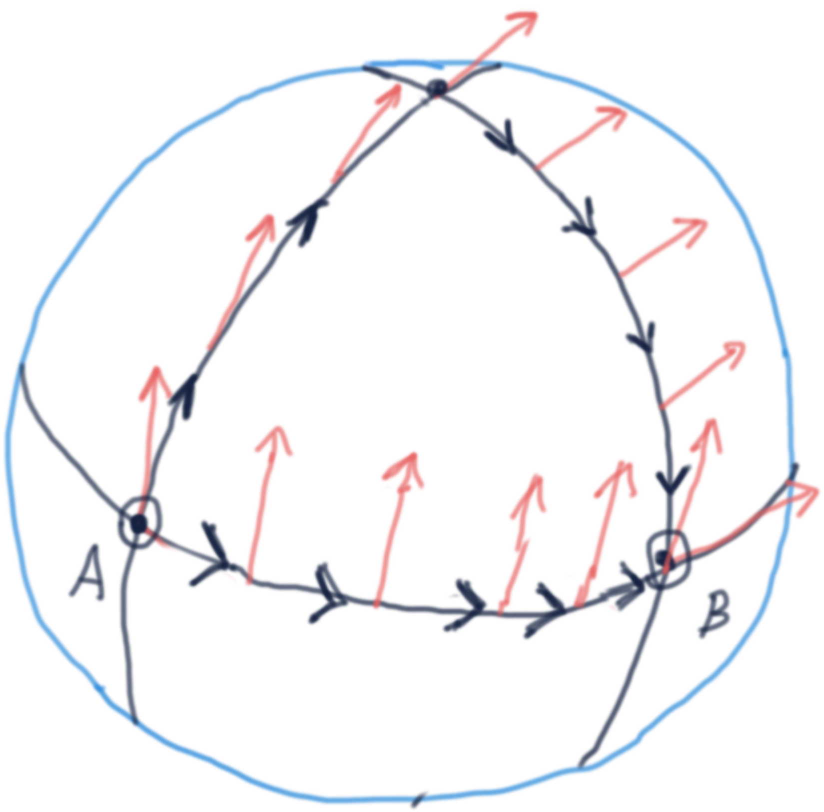
parallel transport is path dependent in a curved one!



Curvature

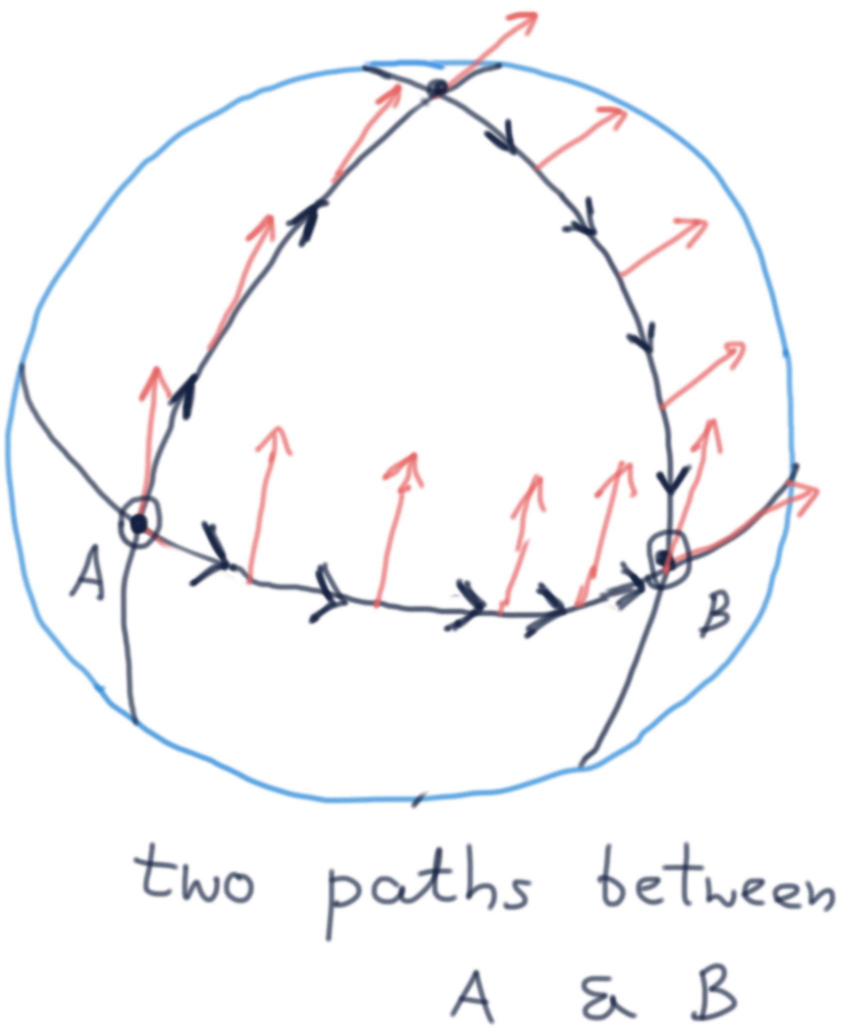
2-dimensional sphere



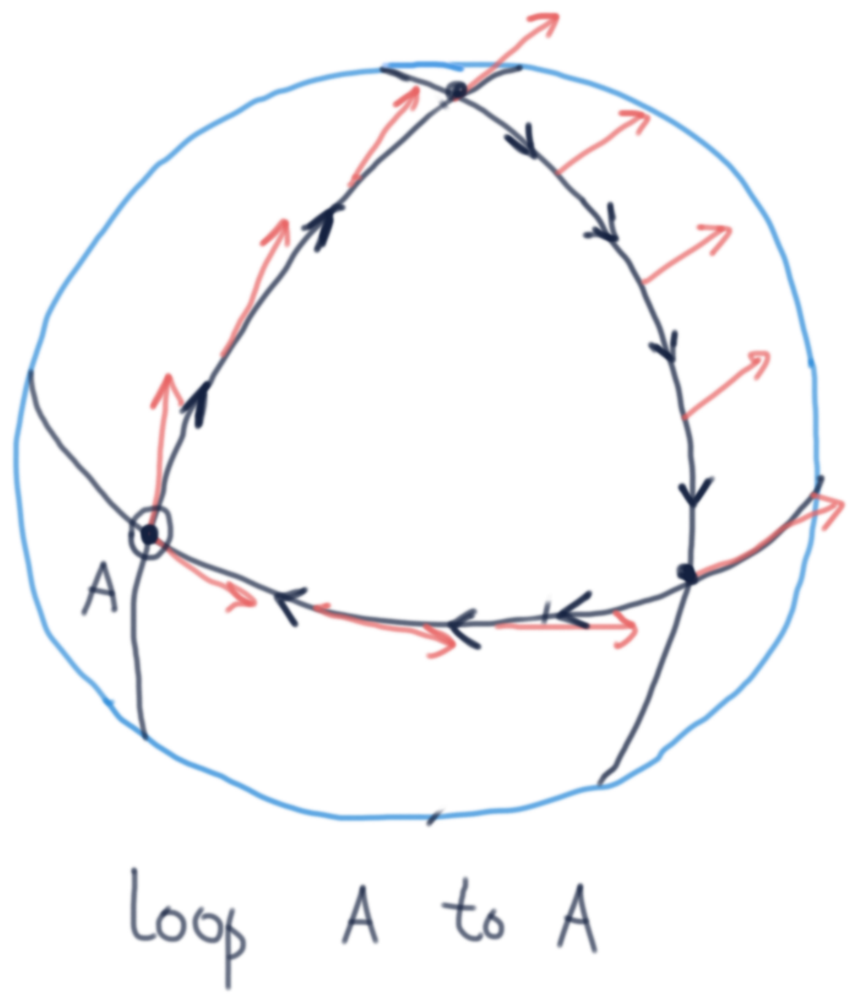
two paths between
A & B

Curvature

2-dimensional sphere



transport over a loop



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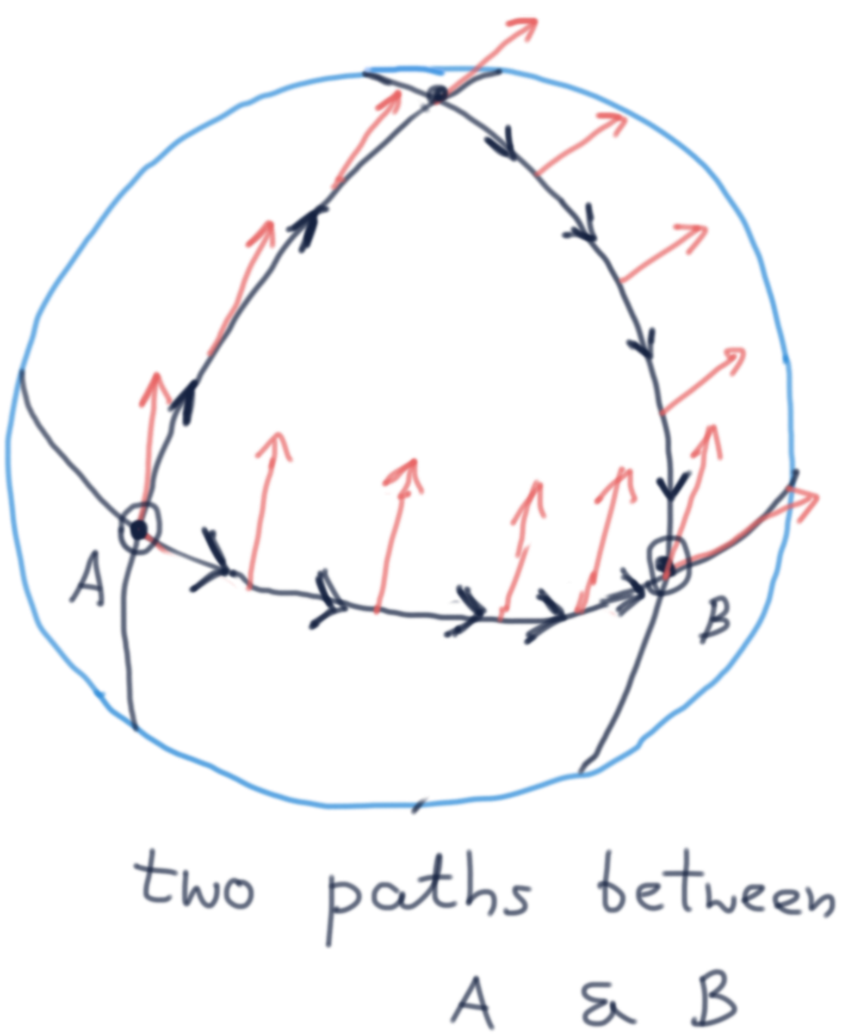

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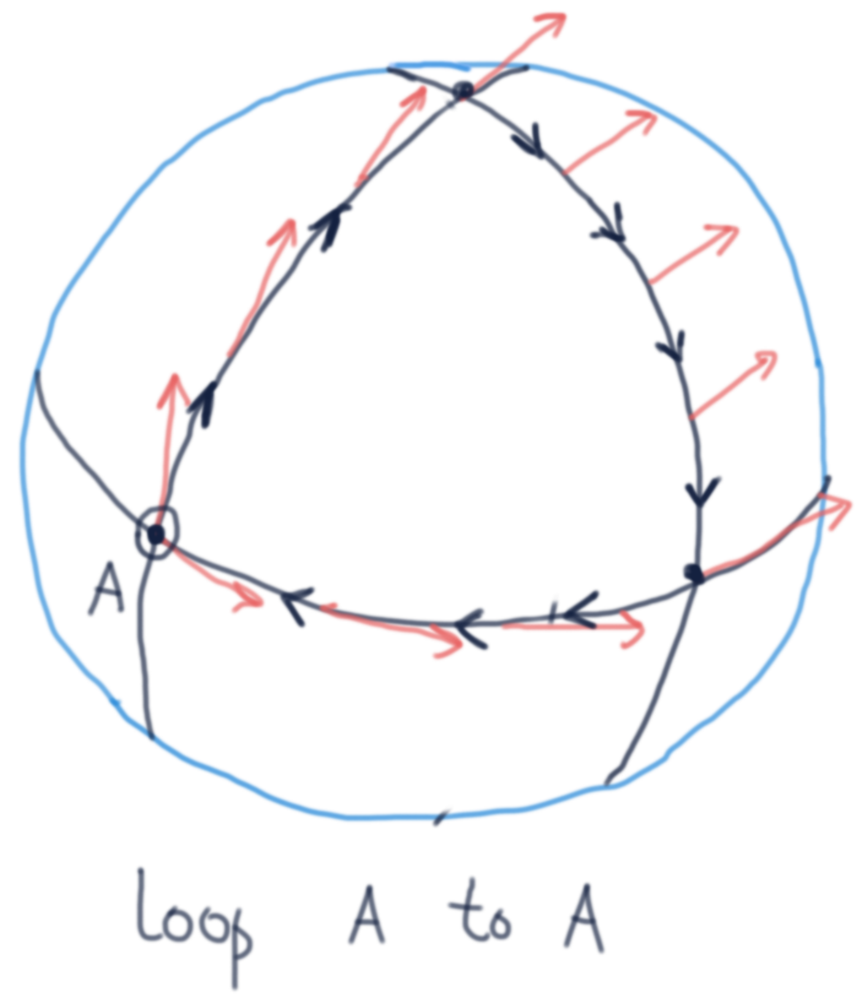

Idea: consider parallel transport over an infinitesimal loop

Curvature

Parallel transport around a loop: fix V_0^μ, X^μ, Y^μ at x_0

$$x^\mu(\sigma) = x_0^\mu + X^\mu \sigma \qquad \sigma \text{ from } 0 \text{ to } \Delta\sigma$$

$$x^\mu(\sigma) = x_0^\mu + X^\mu \Delta\sigma + Y^\mu \rho \qquad \rho \text{ from } 0 \text{ to } \Delta\rho$$

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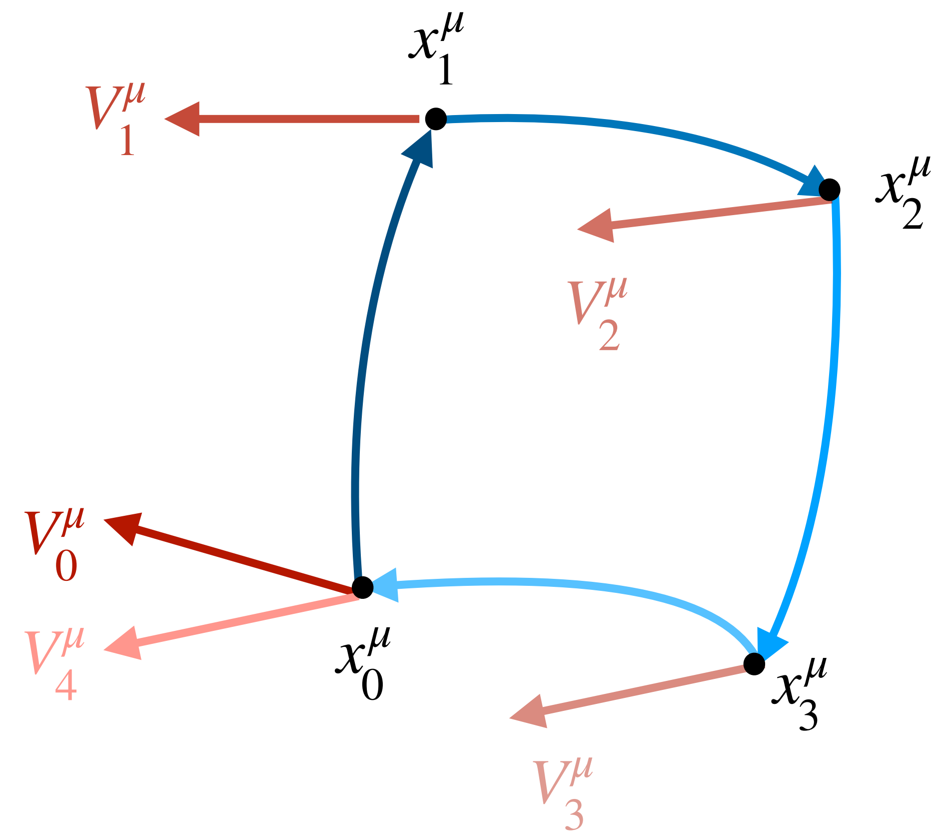

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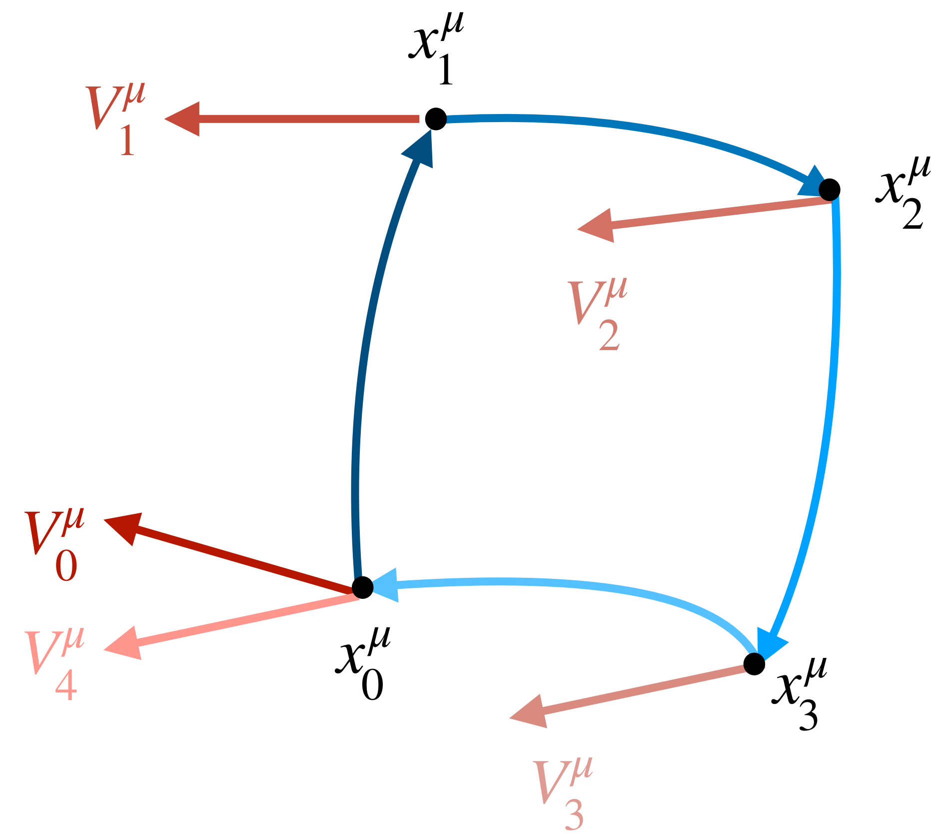
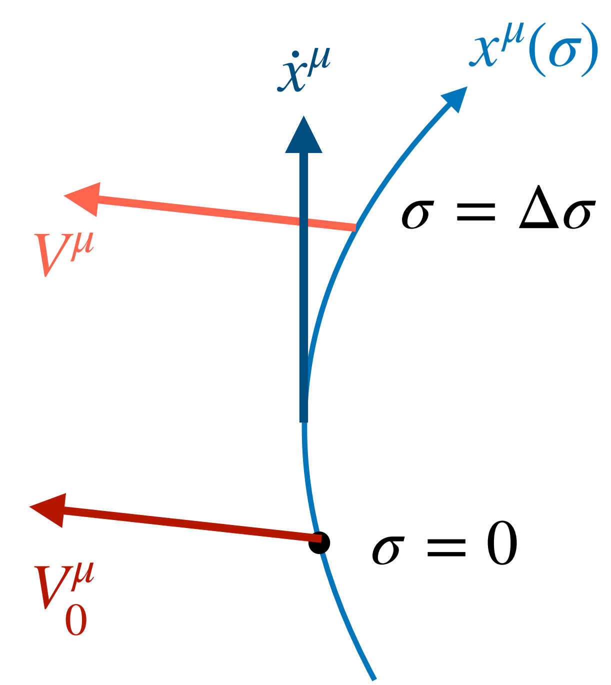


Parallel transport ODE

$$\frac{d}{d\sigma}V^\mu = -\,\Gamma^\mu{}_{\alpha\beta}(x^\kappa(\sigma))\,V^\alpha(\sigma)\,\dot{x}^\beta(\sigma)$$

$$V^\mu(\Delta\sigma) = V_0^\mu - \int_0^{\Delta\sigma} \Gamma^\mu{}_{\alpha\beta}(x^\kappa(\sigma))\,V^\alpha(\sigma)\,\dot{x}^\beta(\sigma)\,d\sigma$$



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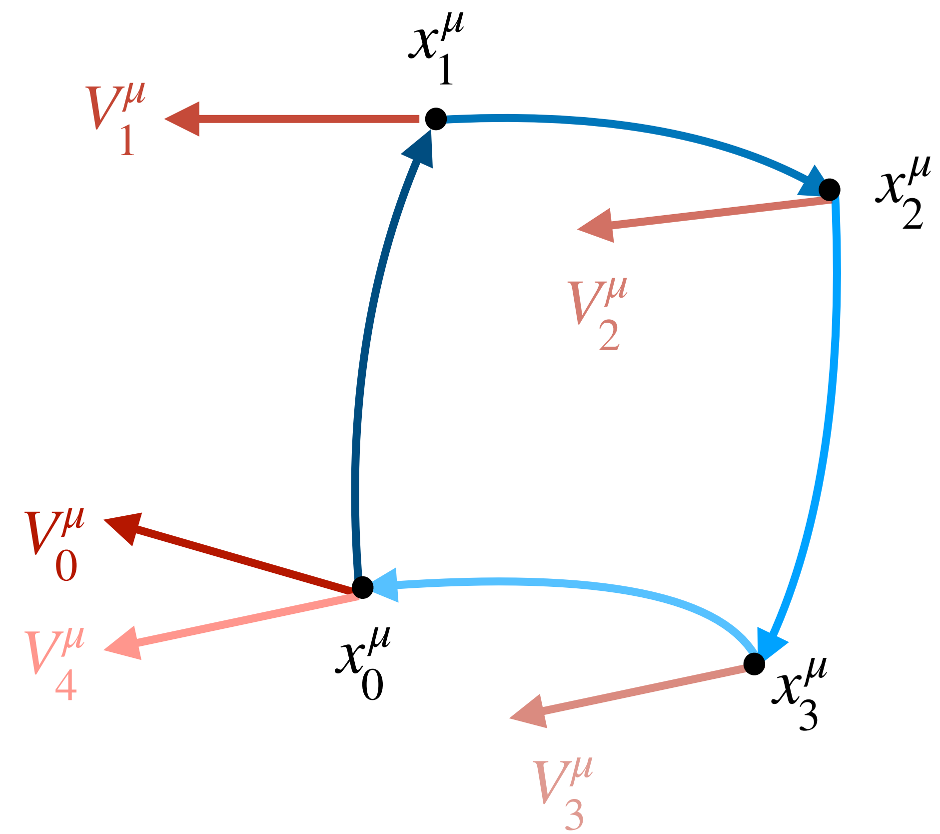

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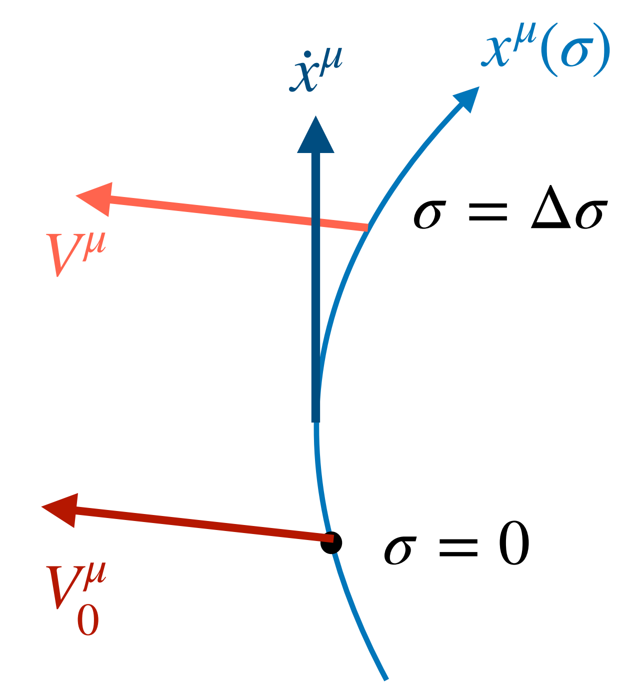


at the second order

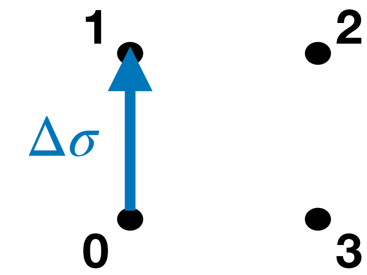
$$V^\mu = V_0^\mu - \Delta\sigma\, \Gamma^\mu{}_{\alpha\beta}(x^\kappa(0))\, V_0^\alpha\, \dot{x}^\beta(0)$$

$$-\frac{\Delta\sigma^2}{2}\left(\Gamma^\mu{}_{\alpha\beta,\gamma}\, V_0^\alpha\, \dot{x}^\beta\, \dot{x}^\gamma + \Gamma^\mu{}_{\sigma\beta}\, \Gamma^\sigma{}_{\alpha\gamma}\, V_0^\alpha\, \dot{x}^\beta\, \dot{x}^\gamma + \Gamma^\mu{}_{\alpha\beta}\, V_0^\alpha\, \ddot{x}^\beta\right)\Bigg|_{\sigma=0} + O(\Delta\sigma^3)$$

Curvature

Curvature

$$V_1^\mu = V_0^\mu - \Gamma^\mu{}_{\alpha\beta}(x_0)\, V_0^\alpha\, X^\beta\, \Delta\sigma + (\,\ldots\,)\, \Delta\sigma^2 + O(\Delta\sigma^3)$$



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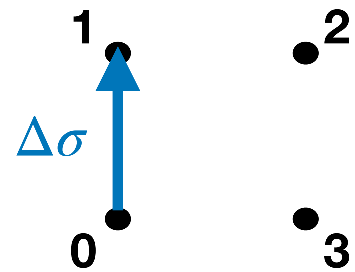
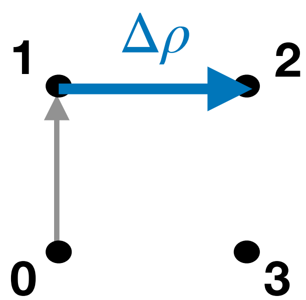


$$V_2^\mu = V_1^\mu - \Gamma^\mu{}_{\alpha\beta}(x_1)\, V_1^\alpha\, Y^\beta\, \Delta\rho + (\ldots)\, \Delta\rho^2 + O(\Delta\rho^3)$$

$$\Gamma^\mu{}_{\alpha\beta}(x_1) = \Gamma^\mu{}_{\alpha\beta}(x_0) + \Gamma^\mu{}_{\alpha\beta,\gamma}(x_0)\, X^\gamma\, \Delta\sigma + O(\Delta\sigma^2)$$



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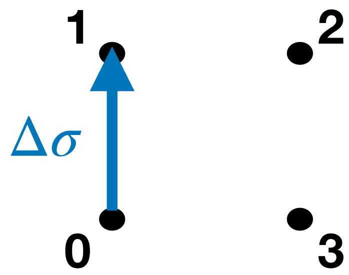

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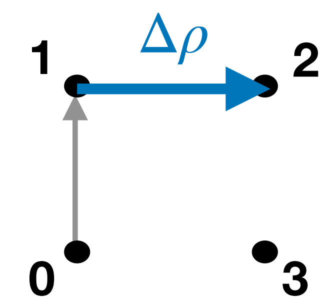
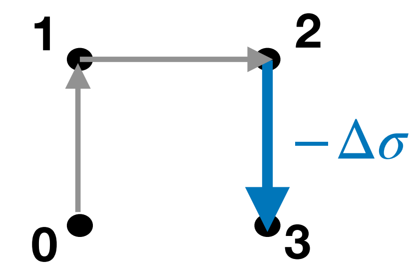


$$V_3^\mu = V_2^\mu + \Gamma^\mu{}_{\alpha\beta}(x_2)\, V_2^\alpha\, X^\beta\, \Delta\sigma + (\,\ldots\,)\,\Delta\sigma^2 + O(\Delta\sigma^3)$$

$$\Gamma^\mu{}_{\alpha\beta}(x_2) = \Gamma^\mu{}_{\alpha\beta}(x_0) + \Gamma^\mu{}_{\alpha\beta,\gamma}(x_0)\, X^\gamma\, \Delta\sigma + \Gamma^\mu{}_{\alpha\beta,\gamma}(x_0)\, Y^\gamma \Delta\rho + O(\Delta\sigma^2, \Delta\rho^2, \Delta\sigma\, \Delta\rho)$$



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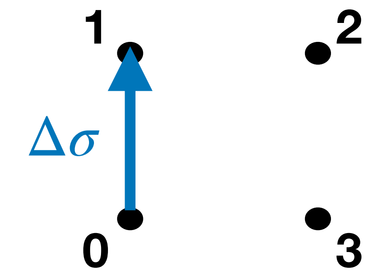

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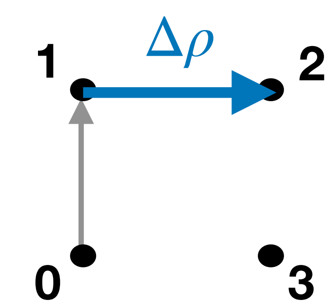

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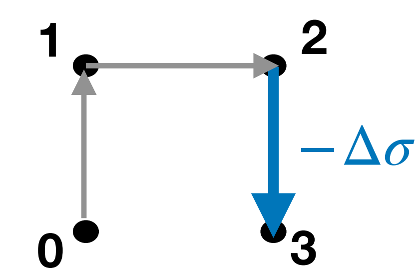
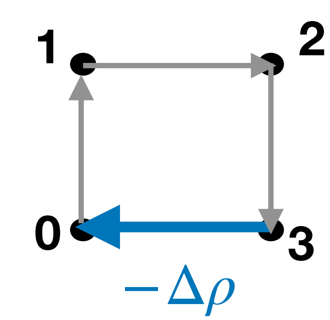


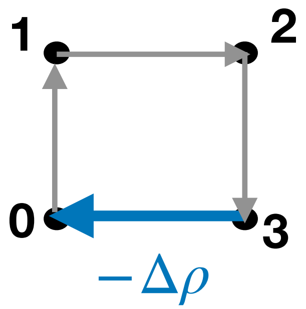
$$V_4^\mu = V_3^\mu + \Gamma^\mu{}_{\alpha\beta}(x_3)\, V_3^\alpha\, Y^\beta\, \Delta\rho + (\,\ldots\,)\,\Delta\rho^2 + O(\Delta\rho^3)$$

$$\Gamma^\mu{}_{\alpha\beta}(x_3) = \Gamma^\mu{}_{\alpha\beta}(x_0) + \Gamma^\mu{}_{\alpha\beta,\gamma}(x_0)\, Y^\gamma \Delta\rho + O(\Delta\sigma^2, \Delta\rho^2, \Delta\sigma\,\Delta\rho)$$



Curvature

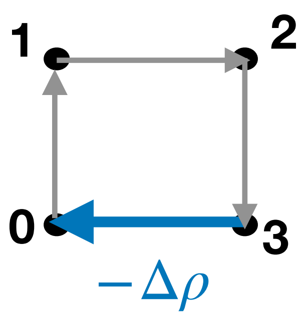
$$V_4^\mu - V_0^\mu = \Delta\rho\,\Delta\sigma \left(\Gamma^\mu{}_{\alpha\beta,\gamma} - \Gamma^\mu{}_{\alpha\gamma,\beta} + \Gamma^\mu{}_{\nu\gamma}\,\Gamma^\nu{}_{\alpha\beta} - \Gamma^\mu{}_{\nu\beta}\,\Gamma^\nu{}_{\alpha\gamma} \right)\Bigg|_{x_0} V_0^\alpha\, X^\beta\, Y^\gamma + O(\Delta^3)$$



This is actually a tensor

Curvature

$$V^\mu_4 - V^\mu_0 = \Delta\rho\,\Delta\sigma\left(\Gamma^\mu{}_{\alpha\beta,\gamma} - \Gamma^\mu{}_{\alpha\gamma,\beta} + \Gamma^\mu{}_{\nu\gamma}\Gamma^\nu{}_{\alpha\beta} - \Gamma^\mu{}_{\nu\beta}\Gamma^\nu{}_{\alpha\gamma}\right)\Big|_{x_0} V^\alpha_0 X^\beta Y^\gamma + O(\Delta^3)$$



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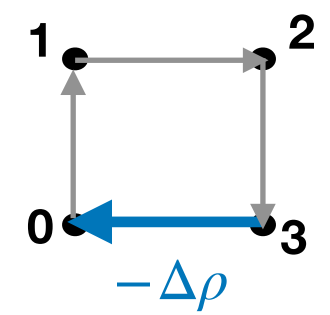
Riemann's curvature tensor: (curvature tensor, Riemann tensor or the Riemann)

$$R^\mu{}_{\nu\alpha\beta} = \partial_\alpha\Gamma^\mu{}_{\nu\beta} - \partial_\beta\Gamma^\mu{}_{\nu\alpha} + \Gamma^\mu{}_{\sigma\alpha}\Gamma^\sigma{}_{\nu\beta} - \Gamma^\mu{}_{\sigma\beta}\Gamma^\sigma{}_{\nu\alpha}$$

Curvature

$$V_4^\mu - V_0^\mu = \Delta\rho\,\Delta\sigma\left(\Gamma^\mu{}_{\alpha\beta,\gamma} - \Gamma^\mu{}_{\alpha\gamma,\beta} + \Gamma^\mu{}_{\nu\gamma}\Gamma^\nu{}_{\alpha\beta} - \Gamma^\mu{}_{\nu\beta}\Gamma^\nu{}_{\alpha\gamma}\right)\Big|_{x_0} V_0^\alpha X^\beta Y^\gamma + O(\Delta^3)$$



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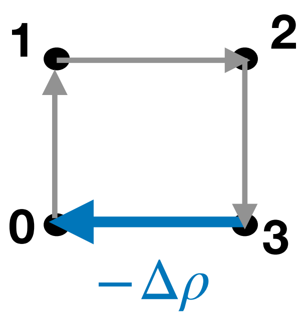
If $R^\mu{}_{\nu\alpha\beta} = 0$ on an open set, then there exist (local) coordinates such that $g_{\mu\nu} = \text{const}$

Curvature

$$V_4^\mu - V_0^\mu = \Delta\rho\,\Delta\sigma\left(\Gamma^\mu{}_{\alpha\beta,\gamma} - \Gamma^\mu{}_{\alpha\gamma,\beta} + \Gamma^\mu{}_{\nu\gamma}\,\Gamma^\nu{}_{\alpha\beta} - \Gamma^\mu{}_{\nu\beta}\,\Gamma^\nu{}_{\alpha\gamma}\right)\Bigg|_{x_0} V_0^\alpha\, X^\beta\, Y^\gamma + O(\Delta^3)$$



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If $R^\mu{}_{\nu\alpha\beta} = 0$ on an open set, then there exist (local) coordinates such that $g_{\mu\nu} = \mathrm{const}$

If $R^\mu{}_{\nu\alpha\beta} = 0$ on an open set, then there the parallel transport is locally independent of path

Curvature

Riemann tensor in locally flat coordinates

In locally flat coordinates ${\Gamma^{\mu}}_{\alpha\beta} = 0,\ \partial_\mu g_{\alpha\beta} = 0$

$${R^{\alpha}}_{\beta\mu\nu} = \partial_\mu {\Gamma^{\alpha}}_{\beta\nu} - \partial_\nu {\Gamma^{\alpha}}_{\beta\mu}$$

Curvature

Riemann tensor in locally flat coordinates

In locally flat coordinates $\Gamma^{\mu}{}_{\alpha\beta} = 0$, $\partial_{\mu} g_{\alpha\beta} = 0$

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}{}_{\beta\nu} - \partial_{\nu}\Gamma^{\alpha}{}_{\beta\mu}$$

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_{\beta} g_{\nu\alpha} + \partial_{\alpha} g_{\nu\beta} - \partial_{\nu} g_{\alpha\beta} \right)$$

$$\partial\Gamma = \frac{1}{2} g^{-1} \left(\partial^2 g + \partial^2 g - \partial^2 g \right) + \frac{1}{2} \partial g^{-1} \left(\partial g + \partial g - \partial g \right)$$

$$\partial_{\alpha}\Gamma^{\mu}{}_{\nu\beta} = \frac{1}{2} g^{\mu\sigma} \left(\partial_{\alpha}\partial_{\nu} g_{\beta\sigma} + \partial_{\alpha}\partial_{\beta} g_{\nu\sigma} - \partial_{\alpha}\partial_{\sigma} g_{\nu\beta} \right)$$

Curvature

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$$\partial_{\alpha}\Gamma^{\mu}{}_{\nu\beta} = \frac{1}{2} g^{\mu\sigma}\left(\partial_{\alpha}\partial_{\nu} g_{\beta\sigma} + \partial_{\alpha}\partial_{\beta} g_{\nu\sigma} - \partial_{\alpha}\partial_{\sigma} g_{\nu\beta}\right)$$

$$R^{\alpha}{}_{\beta\mu\nu} = \frac{1}{2}\, g^{\alpha\sigma}\left(\partial_{\beta}\partial_{\mu} g_{\sigma\nu} - \partial_{\beta}\partial_{\nu} g_{\sigma\mu} + \partial_{\sigma}\partial_{\nu} g_{\beta\mu} - \partial_{\sigma}\partial_{\mu} g_{\beta\nu}\right)$$

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(\partial_{\beta}\partial_{\mu} g_{\alpha\nu} - \partial_{\beta}\partial_{\nu} g_{\alpha\mu} + \partial_{\alpha}\partial_{\nu} g_{\beta\mu} - \partial_{\alpha}\partial_{\mu} g_{\beta\nu}\right)$$

Curvature

Properties of the Riemann

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}$$

All easy to prove in a locally flat frame.
But they are all tensorial, so valid in any coordinates.

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$$

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$$

$$R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0 \qquad \Longleftrightarrow R_{\alpha[\beta\mu\nu]} = 0$$

$$A_{[\alpha\beta\gamma]} = \frac{1}{6}\left(A_{\alpha\beta\gamma} + A_{\beta\gamma\alpha} + A_{\gamma\alpha\beta} - A_{\alpha\gamma\beta} - A_{\beta\alpha\gamma} - A_{\gamma\beta\alpha}\right)$$

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(\partial_\beta\partial_\mu g_{\alpha\nu} - \partial_\beta\partial_\nu g_{\alpha\mu} + \partial_\alpha\partial_\nu g_{\beta\mu} - \partial_\alpha\partial_\mu g_{\beta\nu}\right)$$

20 independent components in dimension 4

Curvature

Commutation of covariant derivatives

$$\nabla_\mu \nabla_\nu f - \nabla_\nu \nabla_\mu f = 0$$ scalar

Curvature

Commutation of covariant derivatives

$$\nabla_\mu \nabla_\nu f - \nabla_\nu \nabla_\mu f = 0$$

scalar

$$\nabla_\mu \nabla_\nu X^\alpha - \nabla_\nu \nabla_\mu X^\alpha = R^\alpha{}_{\beta\mu\nu} X^\beta$$

vector

Curvature

Commutation of covariant derivatives

$$\nabla_\mu \nabla_\nu f - \nabla_\nu \nabla_\mu f = 0$$ scalar

$$\nabla_\mu \nabla_\nu X^\alpha - \nabla_\nu \nabla_\mu X^\alpha = R^\alpha{}_{\beta\mu\nu} X^\beta$$ vector

$$\nabla_\mu \nabla_\nu \kappa_\alpha - \nabla_\nu \nabla_\mu \kappa_\alpha = - R^\beta{}_{\alpha\mu\nu} \kappa_\beta$$ co-vector

Curvature

Commutation of covariant derivatives

$$\nabla_\mu \nabla_\nu f - \nabla_\nu \nabla_\mu f = 0$$ scalar

$$\nabla_\mu \nabla_\nu X^\alpha - \nabla_\nu \nabla_\mu X^\alpha = R^\alpha{}_{\beta\mu\nu} X^\beta$$ vector

$$\nabla_\mu \nabla_\nu \kappa_\alpha - \nabla_\nu \nabla_\mu \kappa_\alpha = -R^\beta{}_{\alpha\mu\nu} \kappa_\beta$$ co-vector

general tensor:

$$\nabla_\alpha \nabla_\beta T^{\mu\nu\ldots}{}_{\kappa\lambda\ldots} - \nabla_\beta \nabla_\alpha T^{\mu\nu\ldots}{}_{\kappa\lambda\ldots} = R^\mu{}_{\sigma\alpha\beta} T^{\sigma\nu\ldots}{}_{\kappa\lambda\ldots} + R^\nu{}_{\sigma\alpha\beta} T^{\mu\sigma\ldots}{}_{\kappa\lambda\ldots} + \cdots$$
$$-R^\sigma{}_{\kappa\alpha\beta} T^{\mu\nu\ldots}{}_{\sigma\lambda\ldots} - R^\sigma{}_{\lambda\alpha\beta} T^{\mu\nu\ldots}{}_{\kappa\sigma\ldots} - \cdots$$

Curvature

Bianchi identities

$$R = \partial\Gamma - \partial\Gamma + \Gamma\,\Gamma - \Gamma\,\Gamma$$

$$\partial R = \partial^2\Gamma - \partial^2\Gamma + \partial\Gamma\,\Gamma + \Gamma\,\partial\Gamma - \partial\Gamma\,\Gamma - \Gamma\,\partial\Gamma$$

in locally flat coordinates $\partial R = \partial^2\Gamma - \partial^2\Gamma$

Curvature

Bianchi identities

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in locally flat coordinates $\partial R = \partial^2\Gamma - \partial^2\Gamma$

$$\Gamma = \frac{1}{2}g^{-1}\left(\partial g + \partial g - \partial g\right)$$

$$\partial\Gamma = \frac{1}{2}g^{-1}\left(\partial^2 g + \partial^2 g - \partial^2 g\right) + \frac{1}{2}\partial g^{-1}\left(\partial g + \partial g - \partial g\right)$$

$$\partial^2\Gamma = \frac{1}{2}g^{-1}\left(\partial^3 g + \partial^3 g - \partial^3 g\right) + \partial g^{-1}\left(\partial^2 g + \partial^2 g - \partial^2 g\right) + \frac{1}{2}\partial^2 g^{-1}\left(\partial g + \partial g - \partial g\right)$$

in locally flat coordinates $\partial^2\Gamma = \frac{1}{2}g^{-1}\left(\partial^3 g + \partial^3 g - \partial^3 g\right)$

Curvature

Bianchi identities

$$R = \partial\Gamma - \partial\Gamma + \Gamma\,\Gamma - \Gamma\,\Gamma$$

$$\partial R = \partial^2\Gamma - \partial^2\Gamma + \partial\Gamma\,\Gamma + \Gamma\,\partial\Gamma - \partial\Gamma\,\Gamma - \Gamma\,\partial\Gamma$$

in locally flat coordinates $\partial R = \partial^2\Gamma - \partial^2\Gamma$

$$\Gamma = \frac{1}{2} g^{-1} \left(\partial g + \partial g - \partial g\right)$$

$$\partial\Gamma = \frac{1}{2} g^{-1} \left(\partial^2 g + \partial^2 g - \partial^2 g\right) + \frac{1}{2}\partial g^{-1} \left(\partial g + \partial g - \partial g\right)$$

$$\partial^2\Gamma = \frac{1}{2} g^{-1} \left(\partial^3 g + \partial^3 g - \partial^3 g\right) + \partial g^{-1} \left(\partial^2 g + \partial^2 g - \partial^2 g\right) + \frac{1}{2}\partial^2 g^{-1} \left(\partial g + \partial g - \partial g\right)$$

in locally flat coordinates $\partial^2\Gamma = \frac{1}{2} g^{-1} \left(\partial^3 g + \partial^3 g - \partial^3 g\right)$

$$\partial_\sigma R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(\partial_\sigma\partial_\beta\partial_\mu g_{\alpha\nu} - \partial_\sigma\partial_\beta\partial_\nu g_{\alpha\mu} + \partial_\sigma\partial_\alpha\partial_\nu g_{\beta\mu} - \partial_\sigma\partial_\alpha\partial_\mu g_{\beta\nu}\right)$$

$$\partial_\sigma R_{\alpha\beta\mu\nu} + \partial_\mu R_{\alpha\beta\nu\sigma} + \partial_\nu R_{\alpha\beta\sigma\mu} = 0 \qquad \Longrightarrow \nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\mu R_{\alpha\beta\nu\sigma} + \nabla_\nu R_{\alpha\beta\sigma\mu} = 0$$

Curvature

Contractions of the Riemann tensor

$$R^{\mu}{}_{\beta\mu\nu} \equiv R_{\beta\nu}$$ Ricci tensor

$$R^{\mu}{}_{\mu} = R$$ Ricci scalar

Curvature

Contractions of the Riemann tensor

$$R^{\mu}{}_{\beta\mu\nu} \equiv R_{\beta\nu}$$ Ricci tensor

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Properties

$$R_{\mu\nu} = R_{\nu\mu}$$

$$\nabla_{\mu} R^{\mu}{}_{\nu} = \frac{1}{2} R_{,\nu}$$

Curvature

Contractions of the Riemann tensor

$$R^{\mu}{}_{\beta\mu\nu} \equiv R_{\beta\nu}$$ Ricci tensor

$$R^{\mu}{}_{\mu} = R$$ Ricci scalar

Properties

$$R_{\mu\nu} = R_{\nu\mu}$$

$$\nabla_{\mu} R^{\mu}{}_{\nu} = \frac{1}{2} R_{,\nu}$$

Other contractions of the Riemann

$$0 = R^{\beta}{}_{\beta\mu\nu} = R_{\mu\nu}{}^{\beta}{}_{\beta}$$

$$R_{\alpha\beta} = -R^{\mu}{}_{\alpha\beta\mu} = -R_{\alpha}{}^{\mu}{}_{\mu\beta} = R_{\alpha}{}^{\mu}{}_{\beta\mu}$$

End of lecture 6