(Pseudo-)Riemannian manifolds

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we will try to mimic the flat spacetime!

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There exists only one connection satisfying 1 and 2 (Levi-Civita or metric connection):

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Christoffel symbols

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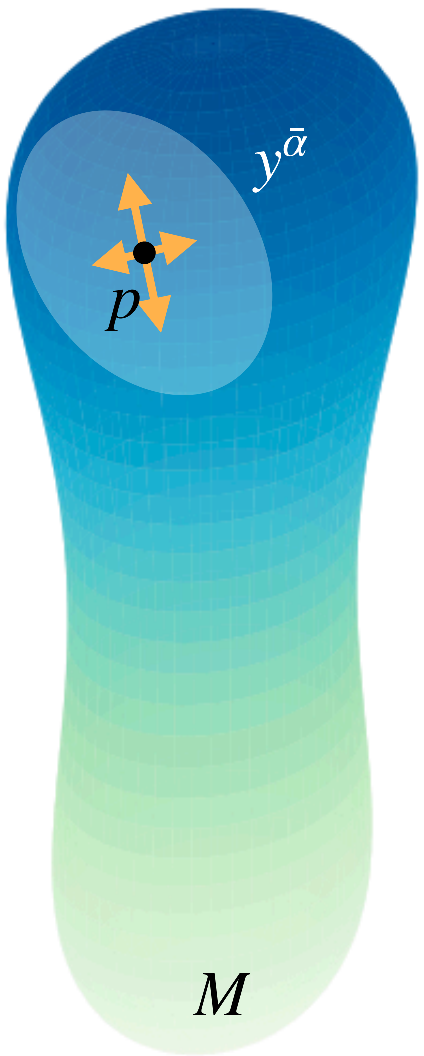
(Pseudo-)Riemannian manifolds

Choosing a connection is equivalent to choosing a class of special coordinate systems at each point



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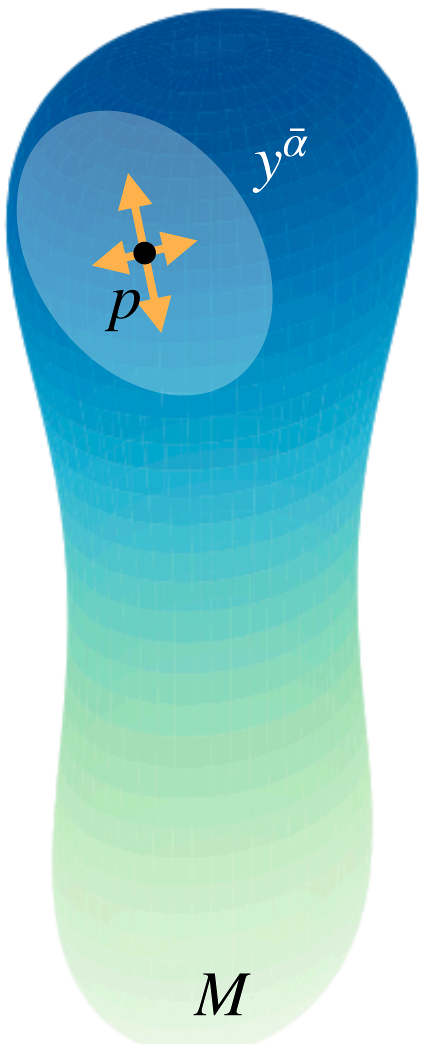

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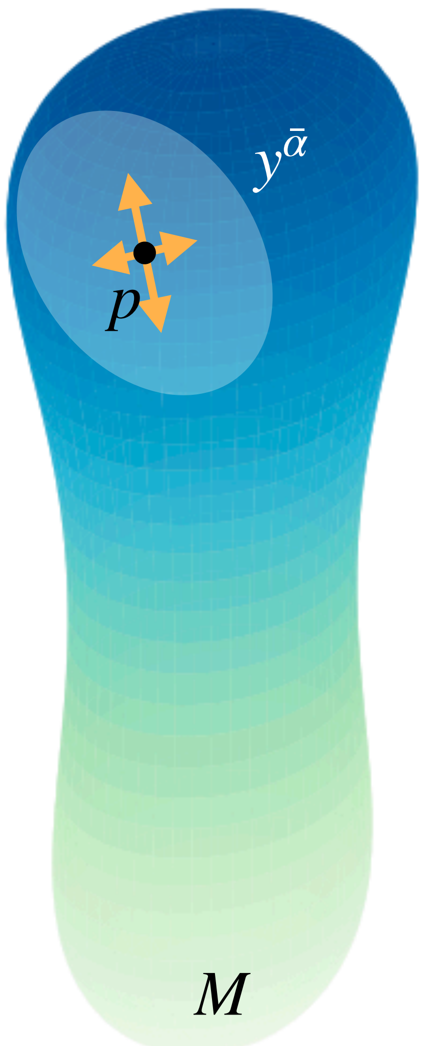

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$$X^{\bar{\mu}}{}_{;\bar{\nu}}(p) = X^{\bar{\mu}}{}_{,\bar{\nu}}(p)$$

„Killing the connection at a single point"

(Pseudo-)Riemannian manifolds

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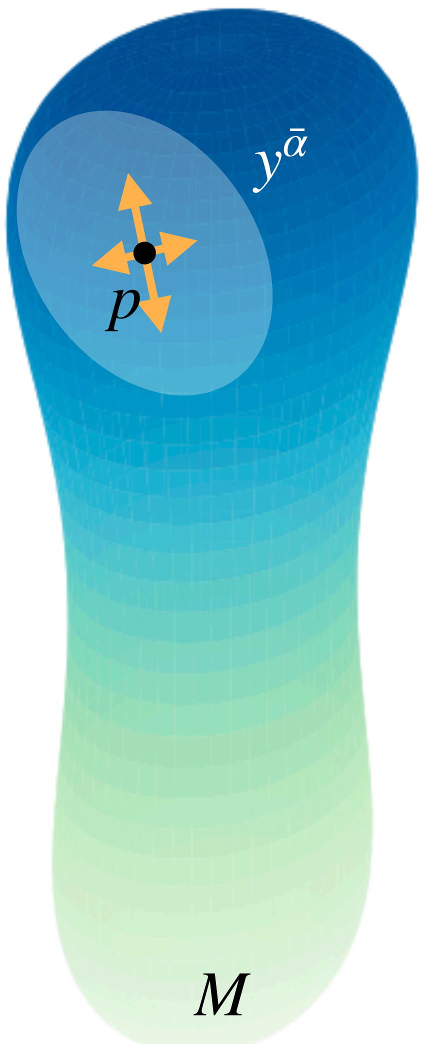

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locally flat coordinates

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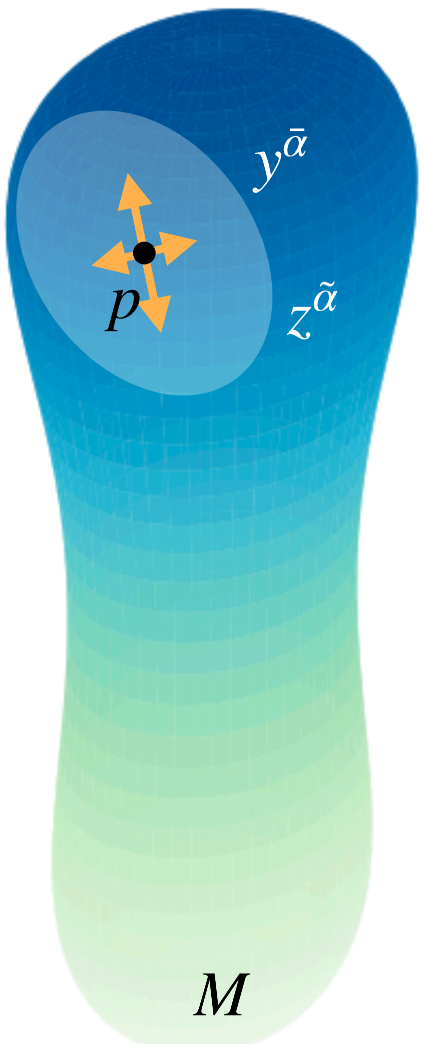
(Pseudo-)Riemannian manifolds

The metric in the new coordinates



For the Levi-Civita connection, the metric in $\left(y^{\bar{\mu}}\right)$ looks like:

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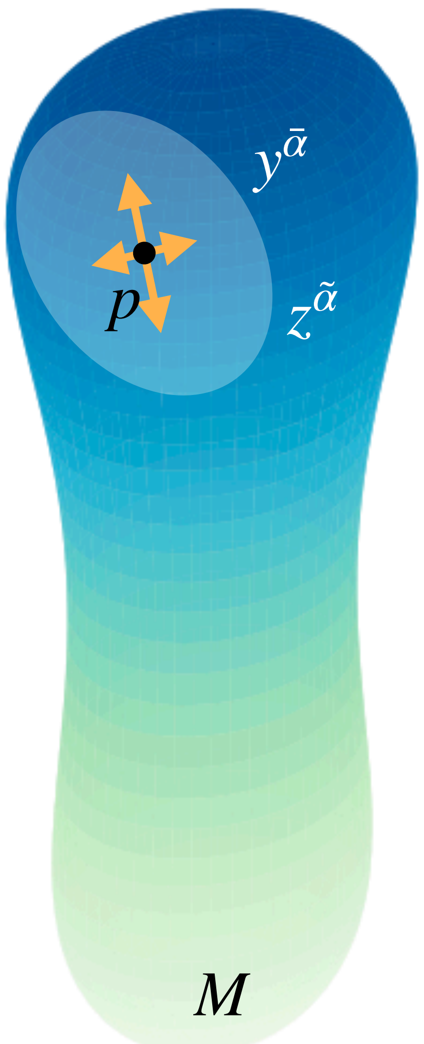

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(Pseudo-)Riemannian manifolds

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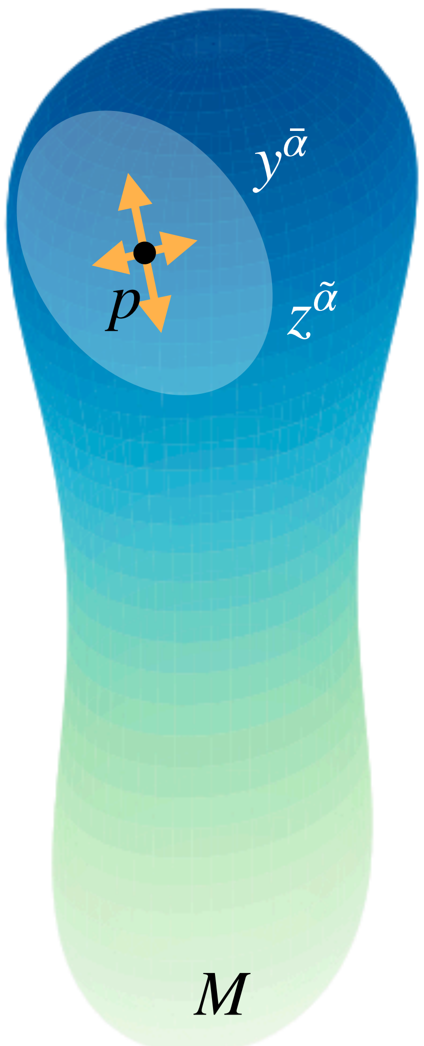

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(Pseudo-)Riemannian manifolds

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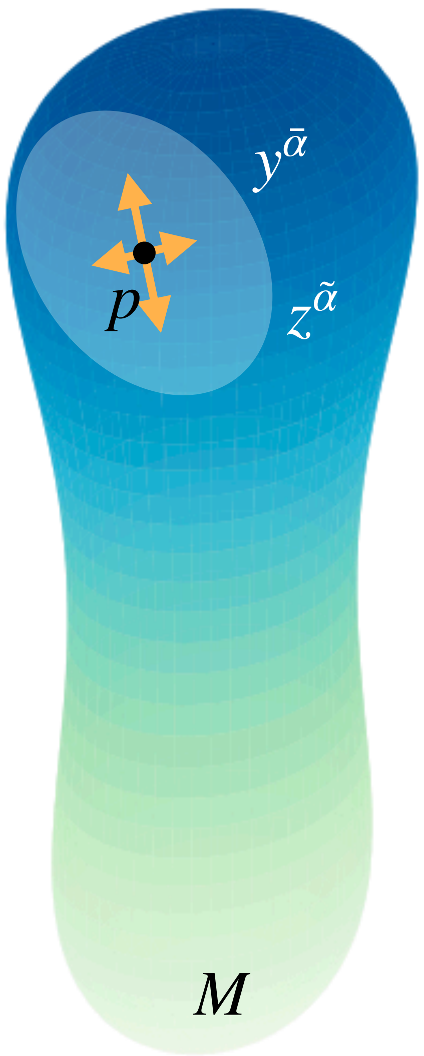

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$$\implies g_{\tilde{\mu}\tilde{\nu}}(z^{\tilde{\sigma}}) = \eta_{\tilde{\mu}\tilde{\nu}} + O(z^2) \qquad X^{\tilde{\mu}}{}_{;\tilde{\nu}}(p) = X^{\tilde{\mu}}{}_{,\tilde{\nu}}(p)$$

Just like a slice of Minkowski (flat) space in Cartesian coordinates!

$\left(z^{\tilde{\mu}}\right)$ **is a great candidate for a local inertial frame at** p

(Pseudo-)Riemannian manifolds

Properties of the covariant derivative (metric connection)

Derivative of a tensor product

$$\nabla_\alpha \left(T^{\mu\nu\cdots}{}_{\rho\sigma\ldots} S^{\kappa\lambda\ldots}{}_{\tau\upsilon\ldots} \right) = \nabla_\alpha T^{\mu\nu\cdots}{}_{\rho\sigma\ldots} S^{\kappa\lambda\ldots}{}_{\tau\upsilon\ldots} + T^{\mu\nu\cdots}{}_{\rho\sigma\ldots} \nabla_\alpha S^{\kappa\lambda\ldots}{}_{\tau\upsilon\ldots}$$

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Commutes with index raising/lowering

given X^μ $$\nabla_\alpha X_\mu = \nabla_\alpha \left(X^\nu g_{\mu\nu} \right) = \left(\nabla_\alpha X^\mu \right) g_{\mu\nu}$$

potentially ambiguous, but not really

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… and with contracting indices

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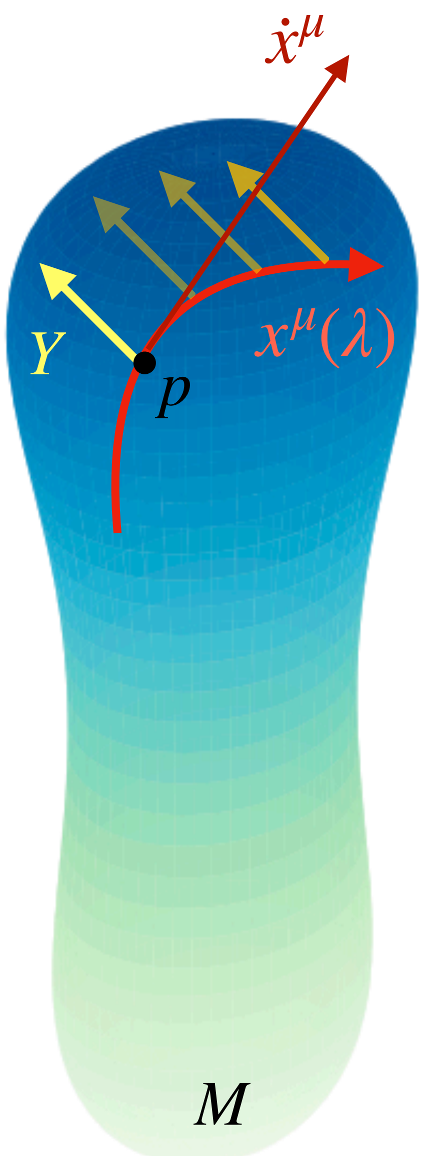
Parallel transport

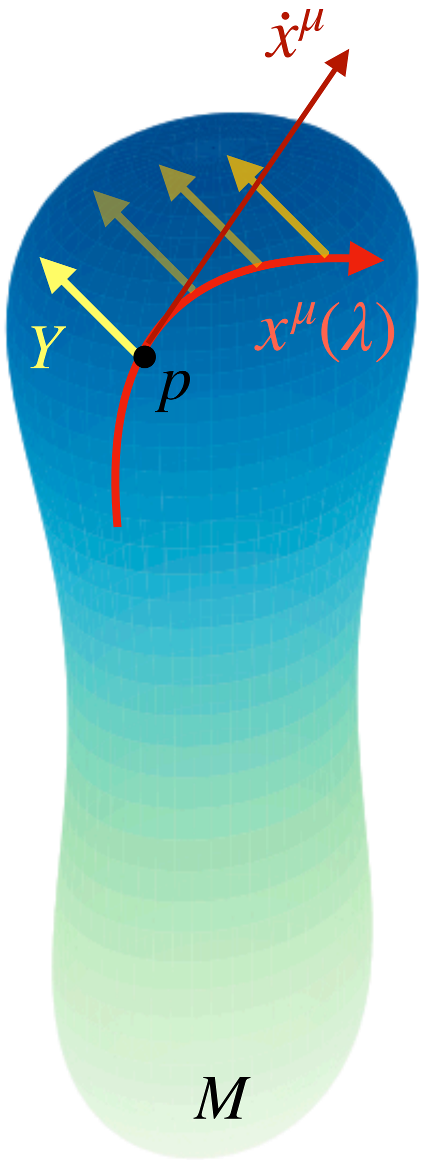


Given a curve through p $\quad x^\mu(\lambda) \qquad\qquad x^\mu(0) = p$

tangent vector $\dot{x}^\mu = \dfrac{dx^\mu}{d\lambda}$

and a vector/tensor Y^μ at p

Parallel transport



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Covariant derivative in direction $\dot{x}^\mu$ or $\dot{x}^\mu \nabla_\mu$

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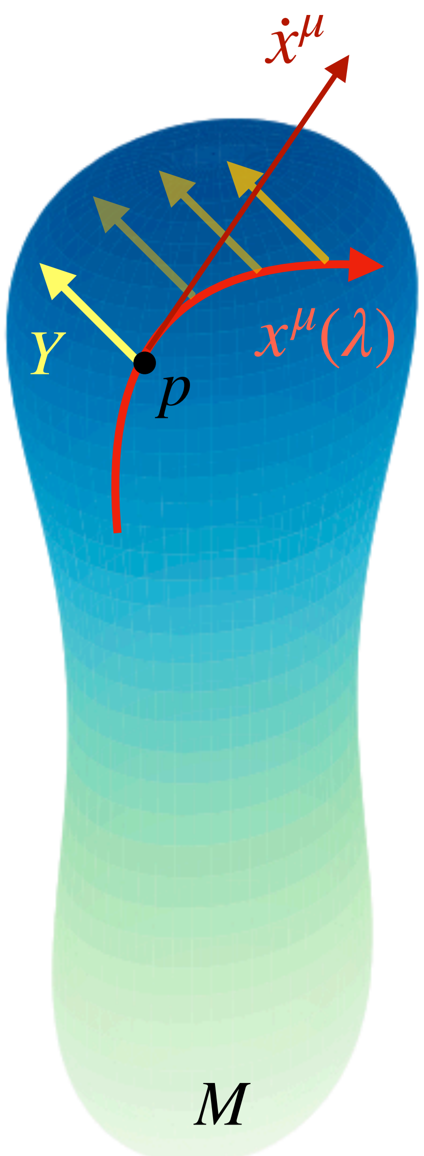

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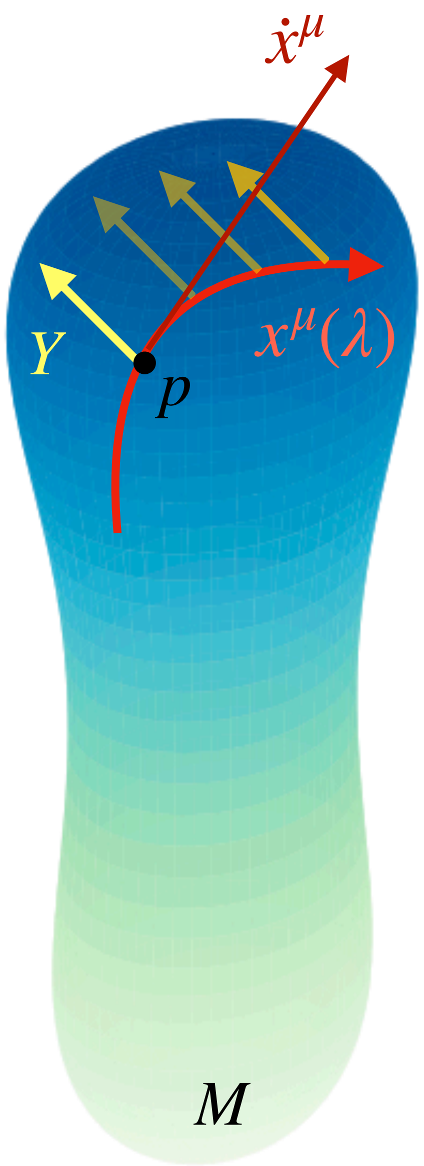
Geometric interpretation: in locally flat coordinates

$$\frac{dY^{\bar{\mu}}}{d\lambda} = 0$$

constant coordinates at linear order

Parallel transport



for general tensors

$$\frac{dT^{\mu\nu\dots}{}_{\alpha\beta\dots}}{d\lambda} + \Gamma^\mu{}_{\sigma\rho} T^{\sigma\nu\dots}{}_{\alpha\beta\dots}\dot{x}^\rho + \Gamma^\nu{}_{\sigma\rho} T^{\mu\sigma\dots}{}_{\alpha\beta\dots}\dot{x}^\rho + \dots - \Gamma^\sigma{}_{\alpha\rho} T^{\mu\nu\dots}{}_{\sigma\beta\dots}\dot{x}^\rho - \Gamma^\sigma{}_{\beta\rho} T^{\mu\nu\dots}{}_{\alpha\sigma\dots}\dot{x}^\rho = 0$$

for the metric $\tilde{g}_{\mu\nu}(\lambda) = g_{\mu\nu}(x^\sigma(\lambda))$

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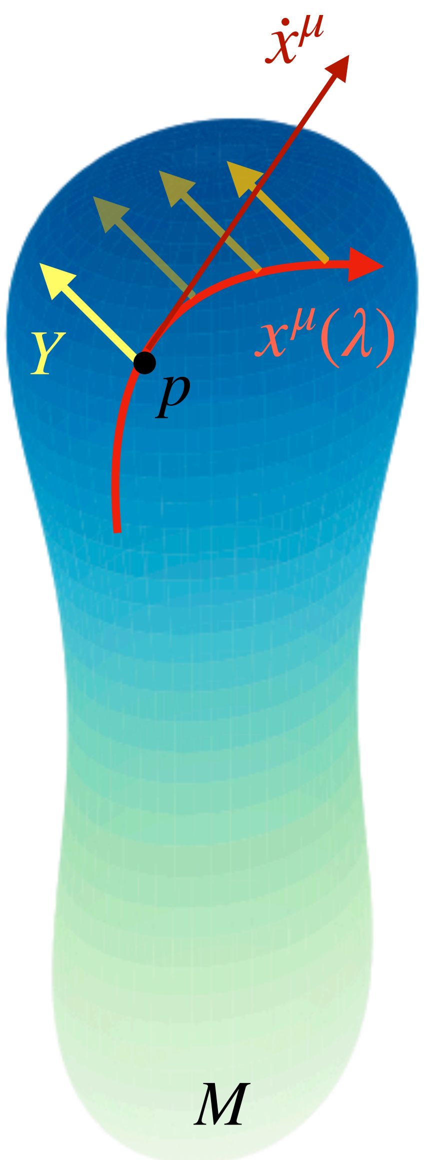

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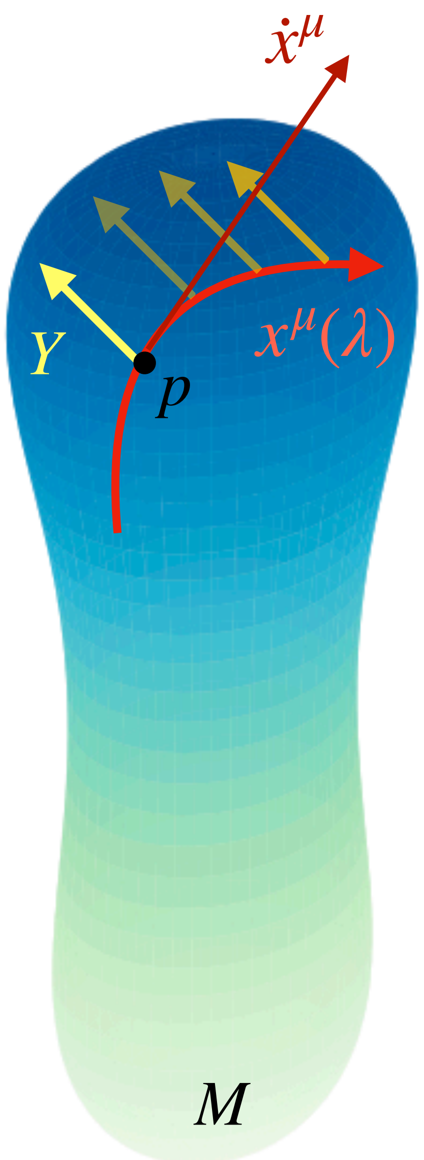

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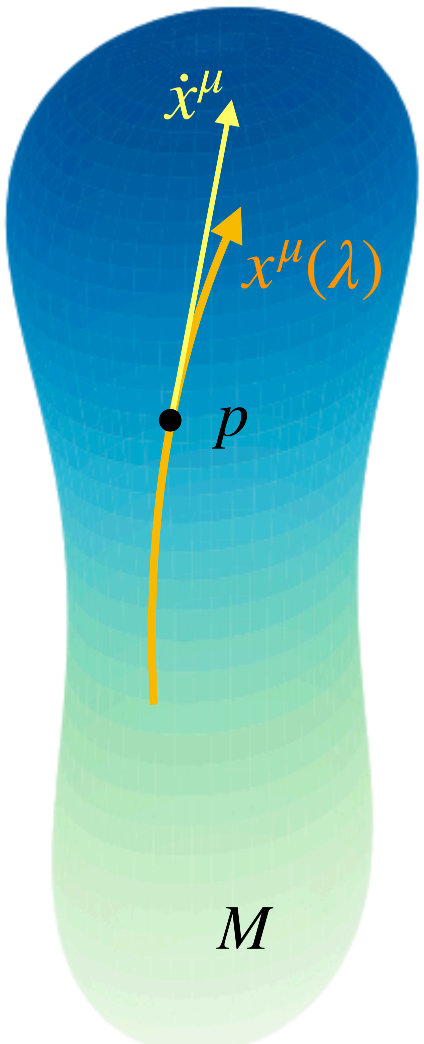
contracting/raising/lowering indices commutes with parallel transport

$$\tilde{X}_\mu(\lambda) = \tilde{X}^\nu(\lambda)\, g_{\mu\nu}(x^\sigma(\lambda)) \qquad \tilde{T}^\mu{}_{\mu\nu}(\lambda) = \tilde{T}^\alpha{}_{\beta\nu}(\lambda)\, \delta^\beta{}_\alpha$$

Geodesics

Special *2n*-parameter family curves defined by the geometry



Curve defined uniquely by a point + tangent vector

Idea: straight line in locally flat coordinates of any point we pass:

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Geodesics

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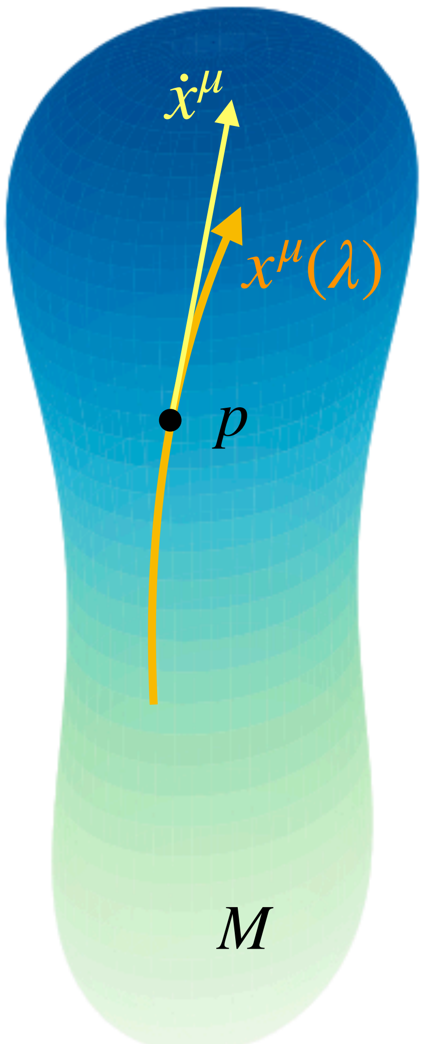

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i.e. the tangent vector is parallel-transported all the time

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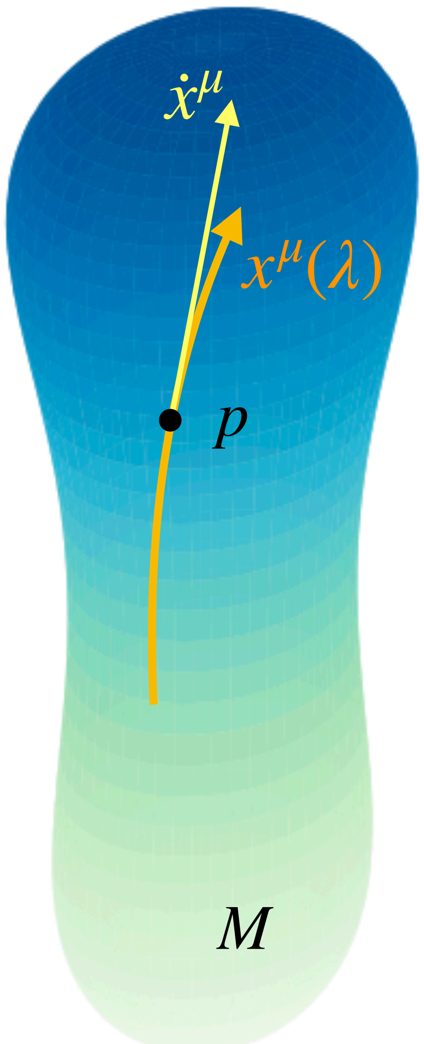

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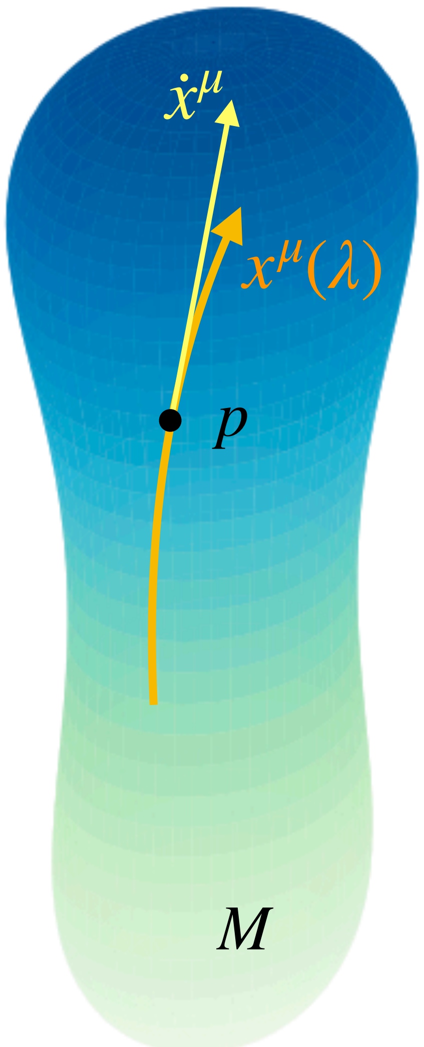
Initial data:

$$x^\mu(0) = x^\mu_p$$

$$\frac{dx^\mu}{d\lambda}(0) = X^\mu_p$$

Geodesics

Properties analogous to straight lines in Minkowski or Euclidean geometry



Conservation of length of the tangent vector $$\frac{d}{d\lambda}\left(\dot{x}^{\mu}\,\dot{x}^{\nu}\,g_{\mu\nu}\right) = 0$$

timelike	null	spacelike
$\dot{x}^{\mu}\,\dot{x}^{\mu}\,g_{\mu\nu} < 0$	$\dot{x}^{\mu}\,\dot{x}^{\mu}\,g_{\mu\nu} = 0$	$\dot{x}^{\mu}\,\dot{x}^{\mu}\,g_{\mu\nu} > 0$

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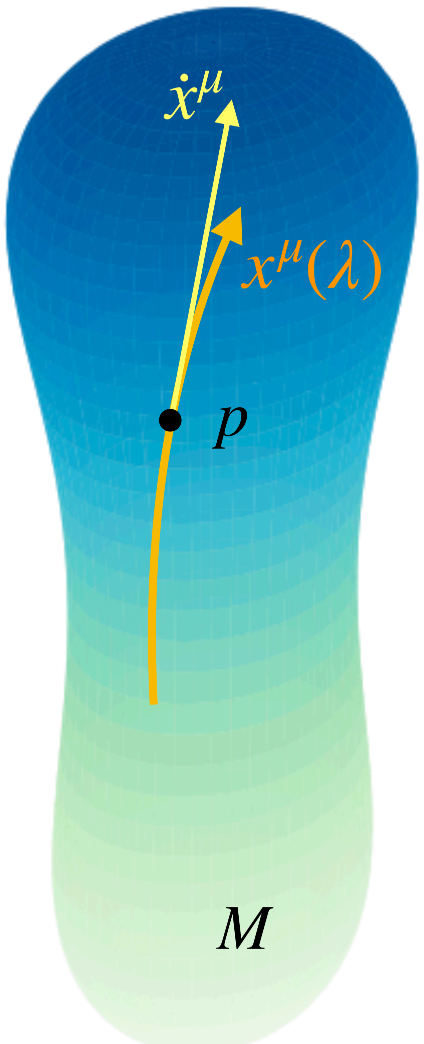

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Reparametrizations by affine transformations

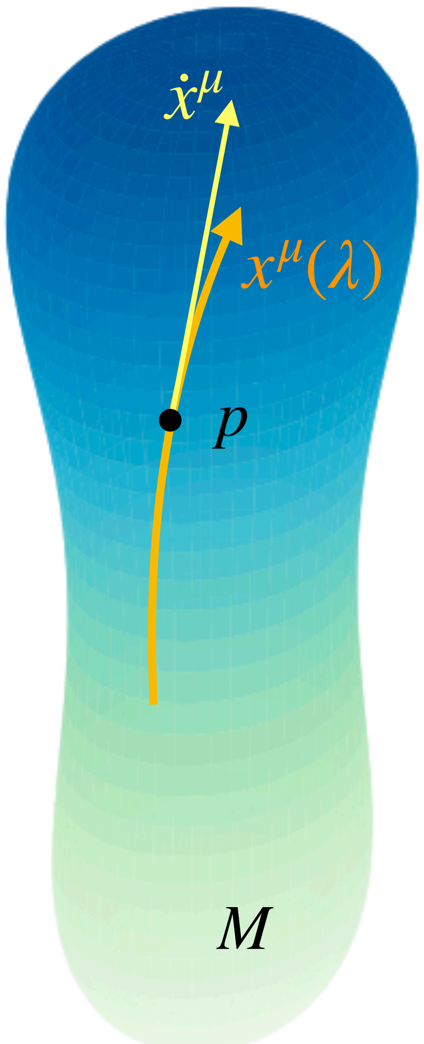
$\lambda \to \lambda' = A\,\lambda + B$ $\qquad A, B = \text{const}$ $\qquad \dot{x}^\mu \to \frac{1}{A}\,\dot{x}^\mu$

If two geodesics share a point p and $\dot{x}_1^\mu\Big|_p = A\,\dot{x}_2^\mu(\lambda)\Big|_p$

then they share the same path, i.e. $x_1^\mu(\lambda) = x_2^\mu\,(A\,\lambda + B)$

Geodesics

Properties analogous to straight lines in Minkowski or Euclidean geometry



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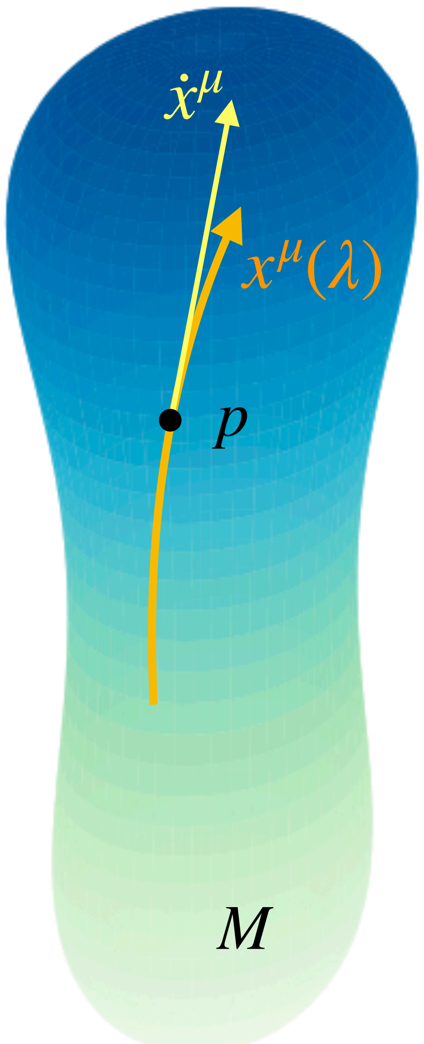
For non-null geodesics we have a preferred parametrization $\dot{x}^\mu\,\dot{x}^\nu\,g_{\mu\nu} = \pm\,1$

Geodesics

Variational principle



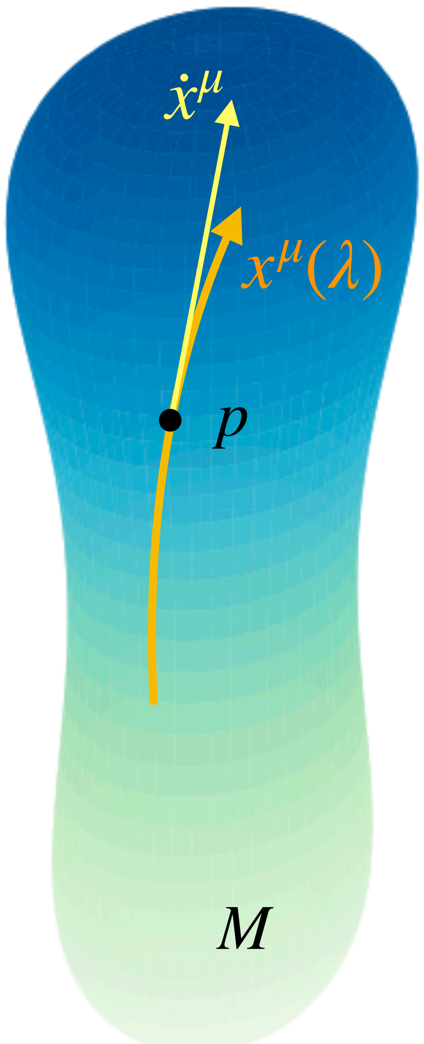
Just like in standard classical mechanics,
the geodesic equation can be derived from a Lagrangian

$$S = \int_{\lambda_0}^{\lambda_1} L(\dot{x}^\mu, x^\mu)\, d\lambda$$

$$L(\dot{x}^\mu, x^\mu) = \frac{1}{2}\, g_{\mu\nu}(x^\alpha)\, \dot{x}^\mu\, \dot{x}^\nu$$

Geodesics

Variational principle



Just like in standard classical mechanics,
the geodesic equation can be derived from a Lagrangian

$$S = \int_{\lambda_0}^{\lambda_1} L(\dot{x}^\mu, x^\mu)\, d\lambda$$

$$L(\dot{x}^\mu, x^\mu) = \frac{1}{2} g_{\mu\nu}(x^\alpha)\, \dot{x}^\mu\, \dot{x}^\nu$$

Fix the initial and final points, vary the curve

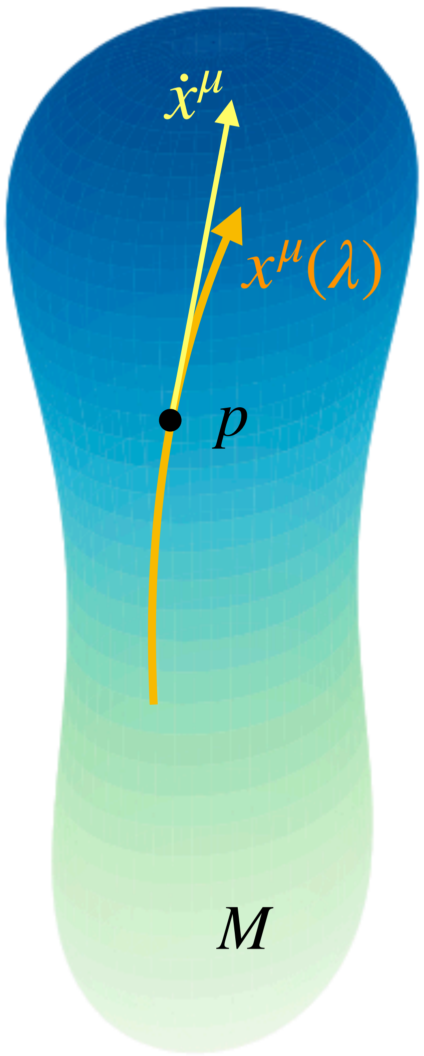
$$x^\mu(\lambda_0) = a^\mu \qquad\qquad x^\mu(\lambda_0) = b^\mu$$

$$\delta S = 0$$

$$\Rightarrow \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$$

Geodesics

Physical interpretation in GR



Timelike geodesics

worldlines of free-falling massive particles

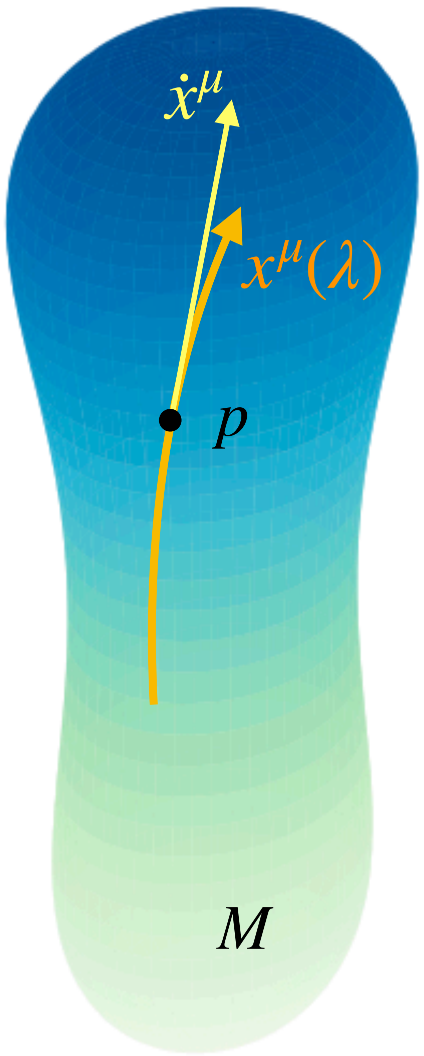
$$\dot{x}^\mu \, \dot{x}_\mu = -1$$

parametrized by proper time τ

$$u^\mu = \dot{x}^\mu = \frac{dx^\mu}{d\tau} \qquad p^\mu = m_0 \, u^\mu$$

Geodesics

Physical interpretation in GR



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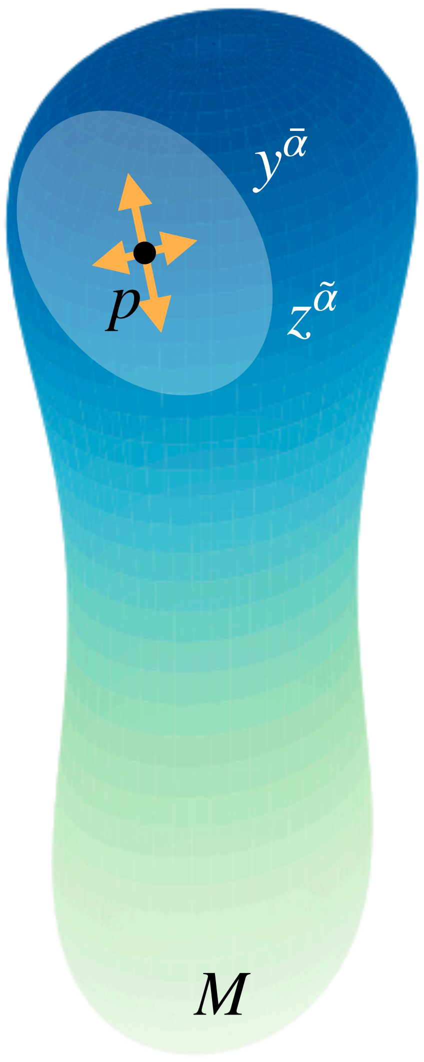
Null geodesics

worldlines of massless particles (photons etc.), light rays

$$\dot{x}^\mu \, \dot{x}_\mu = 0$$

(Pseudo-)Riemannian manifolds



End of lecture 5