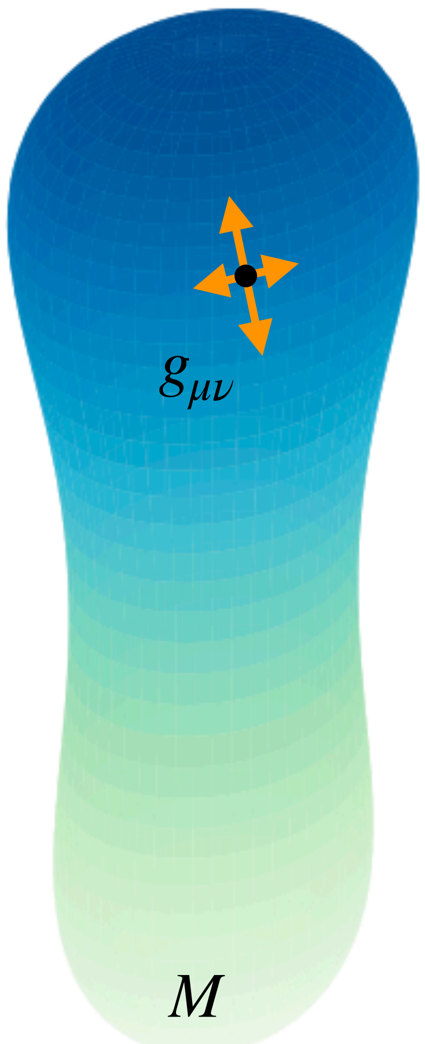
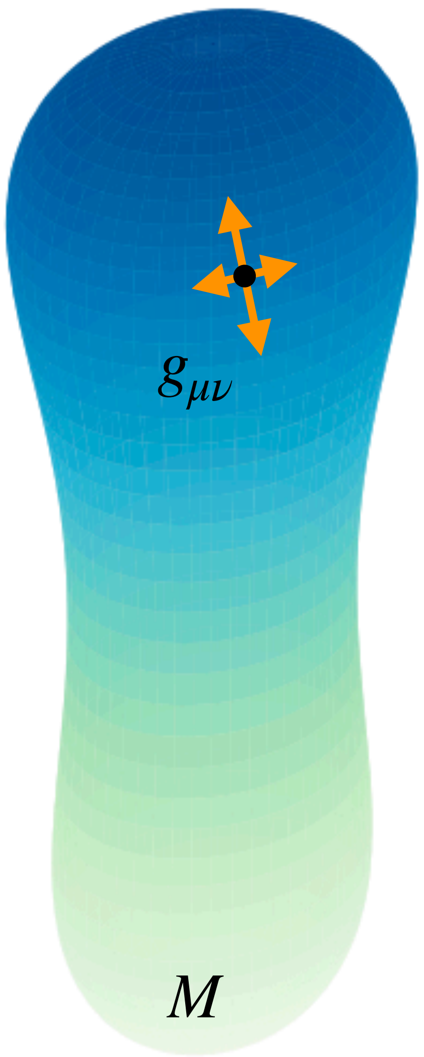
(Pseudo-)Riemannian manifolds



Manifold M

Metric tensor $g_{\mu\nu}(x)$, symmetric non-degenerate

(Pseudo-)Riemannian manifolds



Manifold M

Metric tensor $g_{\mu\nu}(x)$, symmetric non-degenerate

Sylvester's law of inertia

$\Longrightarrow$ we can find a basis in which

$$g_{\mu\nu} = \mathrm{diag}\,(\underbrace{-1, -1, \ldots,}_{k} \underbrace{1, 1, \ldots}_{l})$$

singature (k, l)

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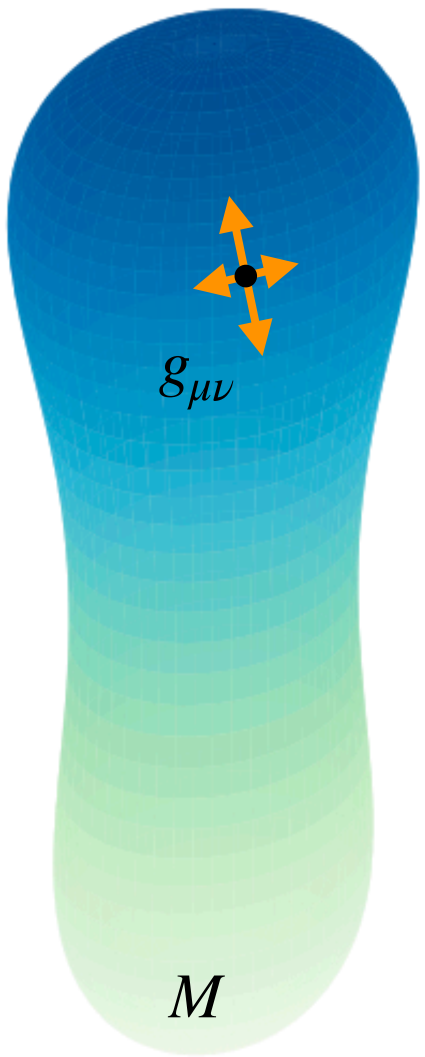

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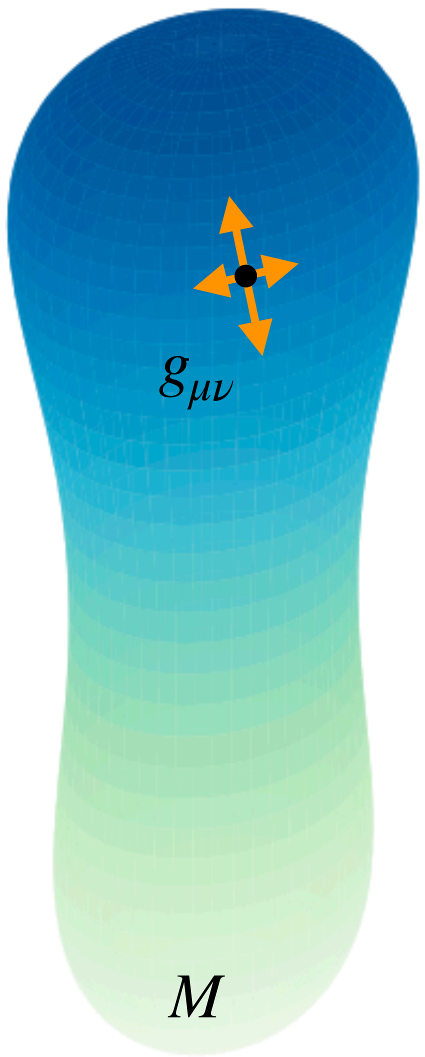
Only pluses $\Rightarrow$ Euclidean signature

$(-1,1,1,1) \Rightarrow$ Lorentzian signature (spacetime)

(Pseudo-)Riemannian manifolds

Connection, covariant derivative, parallel transport



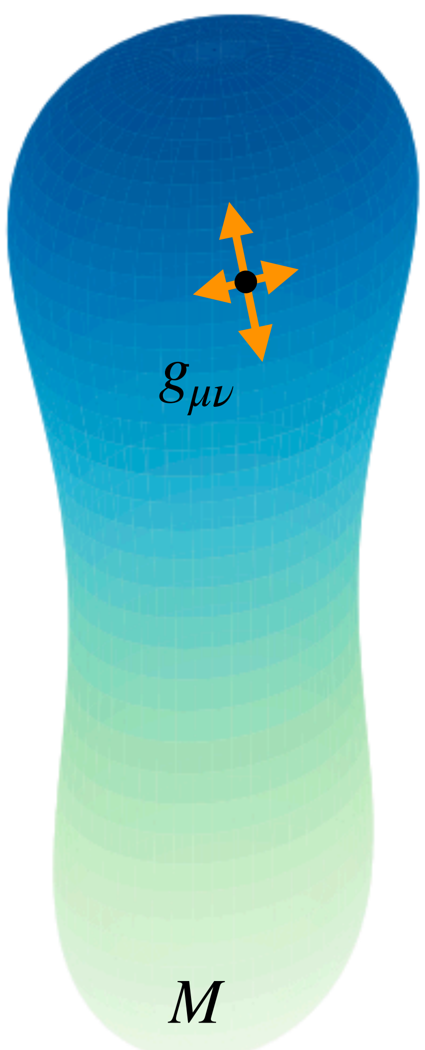
Tensors at different points live in different spaces

We would like to compare tensors at nearby points

Connection = machine for comparing vectors, tensors, 1-forms between infinitesimally close points

(Pseudo-)Riemannian manifolds

Connection, covariant derivative, parallel transport



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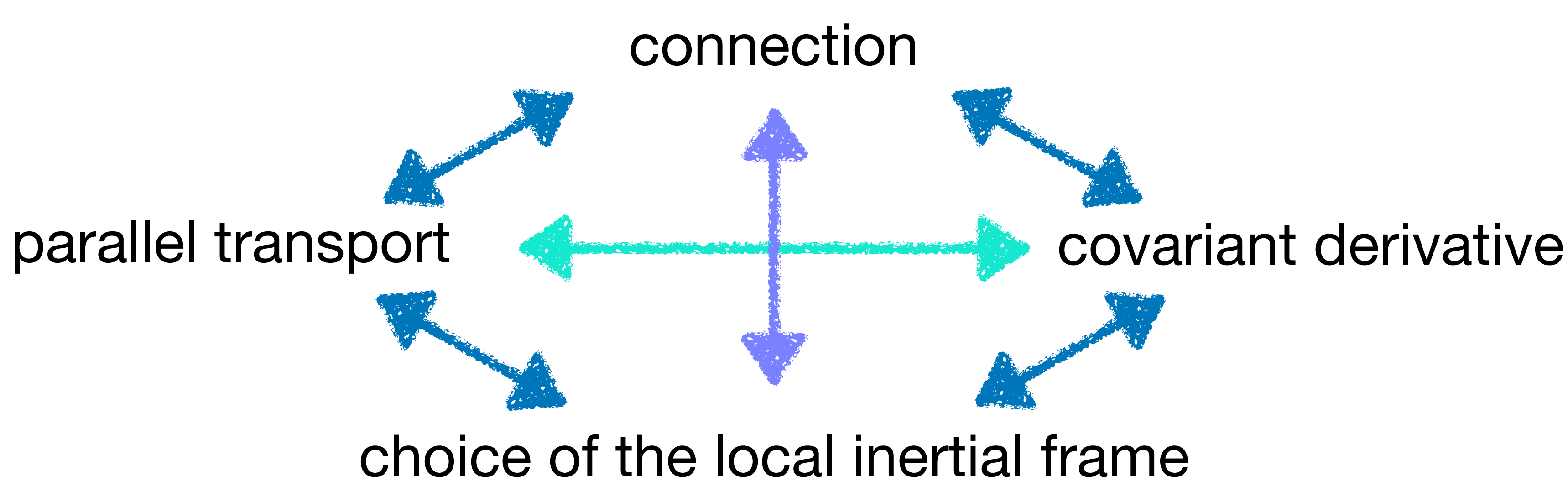

(Pseudo-)Riemannian manifolds

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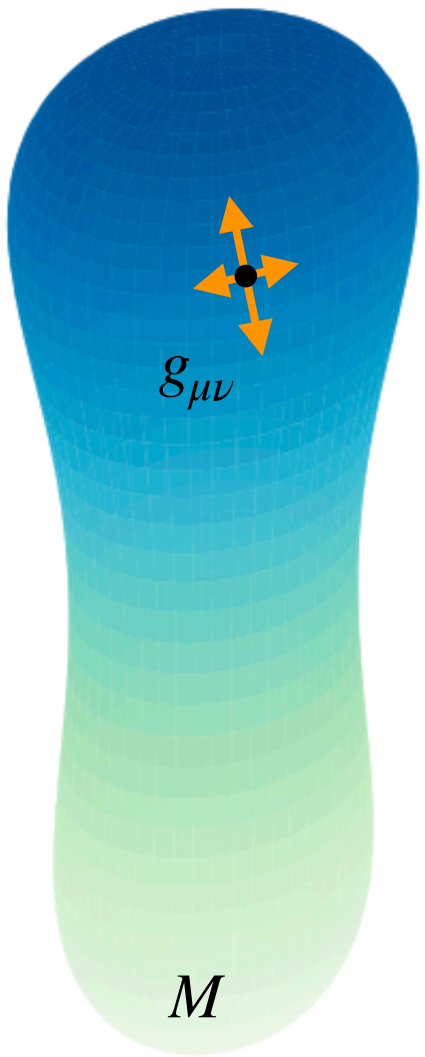

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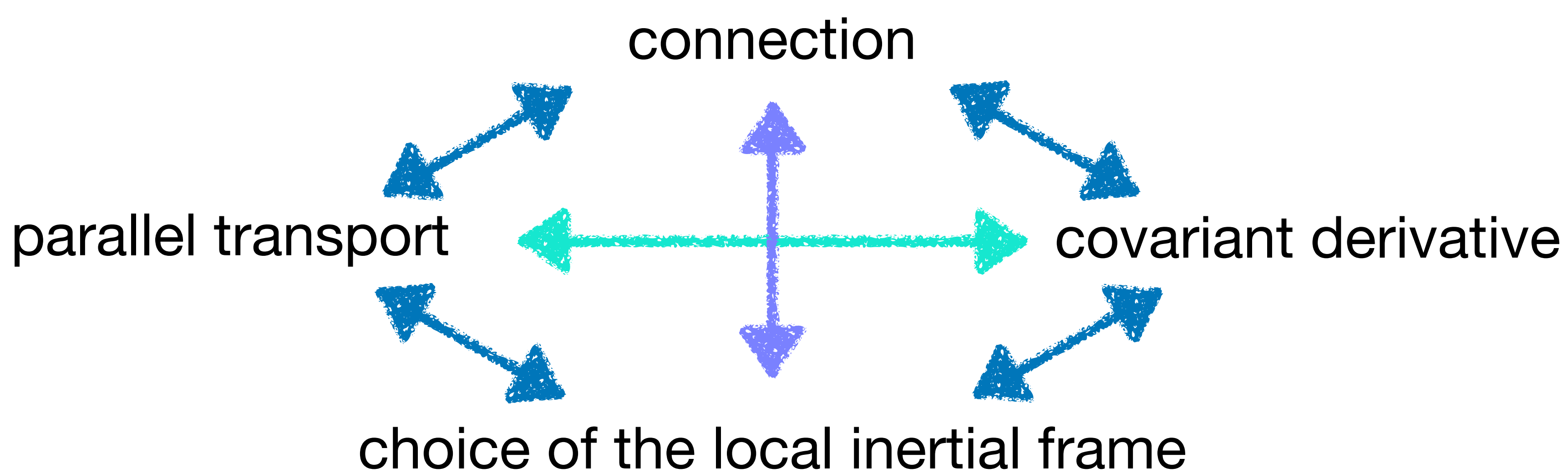

The metric provides a connection called the Levi-Civita/metric connection

(Pseudo-)Riemannian manifolds

How do we choose a connection?

Idea: choose the covariant derivatives of basis vectors

coordinates (x^α) → vector basis $e_{\hat{\mu}}$

coordinates (x^α) → co-vector basis $\omega^{\hat{\nu}}$

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connection coefficients in (x^α)

$$\nabla_\mu e_{\hat{\sigma}} = \Gamma^\nu{}_{\sigma\mu}\, e_{\hat{\nu}}$$

$$\nabla_\mu \omega^{\hat{\nu}} = -\,\Gamma^\nu{}_{\sigma\mu}\, \omega^{\hat{\sigma}}$$

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Covariant derivative component by component

vector

$$\nabla_\alpha X^\mu \equiv X^\mu{}_{;\alpha} = X^\mu{}_{,\alpha} + \Gamma^\mu{}_{\beta\alpha} X^\beta$$

co-vector

$$\nabla_\alpha \kappa_\mu \equiv \kappa_{\mu;\alpha} = \kappa_{\mu,\alpha} - \Gamma^\beta{}_{\mu\alpha} \kappa_\beta$$

scalar

$$f_{;\mu} \equiv f_{,\mu}$$

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$$f_{;\mu} \equiv f_{,\mu}$$

general tensor:

$$\nabla_\alpha T^{\mu\nu\dots}{}_{\kappa\lambda\dots} \equiv T^{\mu\nu\dots}{}_{\kappa\lambda\dots;\alpha} = T^{\mu\nu\dots}{}_{\kappa\lambda\dots,\alpha} + \Gamma^\mu{}_{\beta\alpha}\, T^{\beta\nu\dots}{}_{\kappa\lambda\dots} + \Gamma^\nu{}_{\beta\alpha}\, T^{\mu\beta\dots}{}_{\kappa\lambda\dots} + \cdots$$
$$-\Gamma^\beta{}_{\kappa\alpha}\, T^{\mu\nu\dots}{}_{\beta\lambda\dots} - \Gamma^\beta{}_{\lambda\alpha}\, T^{\mu\nu\dots}{}_{\kappa\beta\dots} - \cdots$$

(Pseudo-)Riemannian manifolds

How do we choose a connection?

Idea: choose the covariant derivatives of basis vectors

connection coefficients in $\left(x^{\alpha}\right)$

coordinates $\left(x^{\alpha}\right)$ → vector basis $e_{\hat{\mu}}$

→ co-vector basis $\omega^{\hat{\nu}}$

$$\nabla_{\mu} e_{\hat{\sigma}} = \Gamma^{\nu}{}_{\sigma\mu}\, e_{\hat{\nu}}$$

$$\nabla_{\mu} \omega^{\hat{\nu}} = -\Gamma^{\nu}{}_{\sigma\mu}\, \omega^{\hat{\sigma}}$$

$$\Gamma^{\mu}{}_{\nu\alpha}(x^{\sigma})$$

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$$- \Gamma^{\beta}{}_{\kappa\alpha}\, T^{\mu\nu\ldots}{}_{\beta\lambda\ldots} - \Gamma^{\beta}{}_{\lambda\alpha}\, T^{\mu\nu\ldots}{}_{\kappa\beta\ldots} - \cdots$$

(Pseudo-)Riemannian manifolds

coordinate change $(x^\mu) \to \left(y^{\bar\mu}\right)$

Jacobian $\Lambda^{\bar\mu}{}_\mu = \dfrac{\partial y^{\bar\mu}}{\partial x^\mu}$ inverse Jacobian $\Lambda^{\mu}{}_{\bar\mu} = \dfrac{\partial x^{\mu}}{\partial y^{\bar\mu}}$

$$\Gamma^\mu{}_{\nu\alpha} \to \Gamma^{\bar\mu}{}_{\bar\nu\bar\alpha} = \Gamma^\mu{}_{\nu\alpha}\,\Lambda^{\bar\mu}{}_\mu\,\Lambda^\nu{}_{\bar\nu}\,\Lambda^\alpha{}_{\bar\alpha} - \Lambda^{\bar\mu}{}_{\nu,\alpha}\,\Lambda^\nu{}_{\bar\nu}\,\Lambda^\alpha{}_{\bar\alpha}$$

not a tensor!

$$\frac{\partial^2 y^{\bar\mu}}{\partial x^\nu\,\partial x^\alpha}$$

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not a tensor!

$$\Lambda^{\bar\mu}{}_{\nu,\alpha} = \frac{\partial^2 y^{\bar\mu}}{\partial x^\nu\,\partial x^\alpha}$$

on the other hand:

$$X^\mu \to X^{\bar\mu} = X^\mu\,\Lambda^{\bar\mu}{}_\mu$$

$$X^\mu{}_{,\nu} \to X^{\bar\mu}{}_{,\bar\nu} = X^\mu{}_{,\nu}\,\Lambda^{\bar\mu}{}_\mu\,\Lambda^\nu{}_{\bar\nu} + X^\mu\,\Lambda^{\bar\mu}{}_{\mu,\nu}\,\Lambda^\nu{}_{\bar\nu}$$

not a tensor!

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$$\Gamma^{\mu}{}_{\nu\alpha} \to \Gamma^{\bar{\mu}}{}_{\bar{\nu}\bar{\alpha}} = \Gamma^{\mu}{}_{\nu\alpha}\, \Lambda^{\bar{\mu}}{}_{\mu}\, \Lambda^{\nu}{}_{\bar{\nu}}\, \Lambda^{\alpha}{}_{\bar{\alpha}} - \Lambda^{\bar{\mu}}{}_{\nu,\alpha}\, \Lambda^{\nu}{}_{\bar{\nu}}\, \Lambda^{\alpha}{}_{\bar{\alpha}}$$

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$$X^{\mu}{}_{,\nu} \to X^{\bar{\mu}}{}_{,\bar{\nu}} = X^{\mu}{}_{,\nu}\, \Lambda^{\bar{\mu}}{}_{\mu}\, \Lambda^{\nu}{}_{\bar{\nu}} + X^{\mu}\, \Lambda^{\bar{\mu}}{}_{\mu,\nu}\, \Lambda^{\nu}{}_{\bar{\nu}}$$

not a tensor!

but magically, if we take both together… $X^{\mu}{}_{;\alpha} = X^{\mu}{}_{,\alpha} + \Gamma^{\mu}{}_{\beta\alpha}\, X^{\beta}$

$$X^{\mu}{}_{;\nu} \to X^{\bar{\mu}}{}_{;\bar{\nu}} = X^{\mu}{}_{;\nu}\, \Lambda^{\bar{\mu}}{}_{\mu}\, \Lambda^{\nu}{}_{\bar{\nu}}$$

this is a tensor!