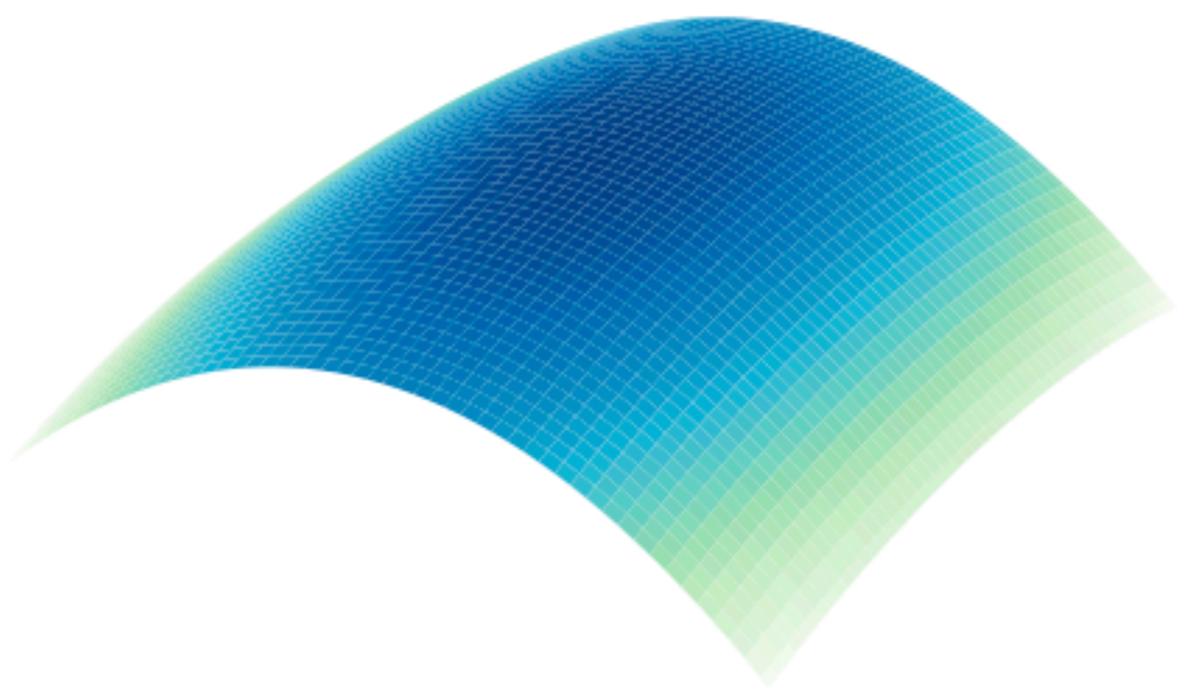
Manifolds and differential geometry

Manifold

generalization of the notion of a (n-dimensional) surface

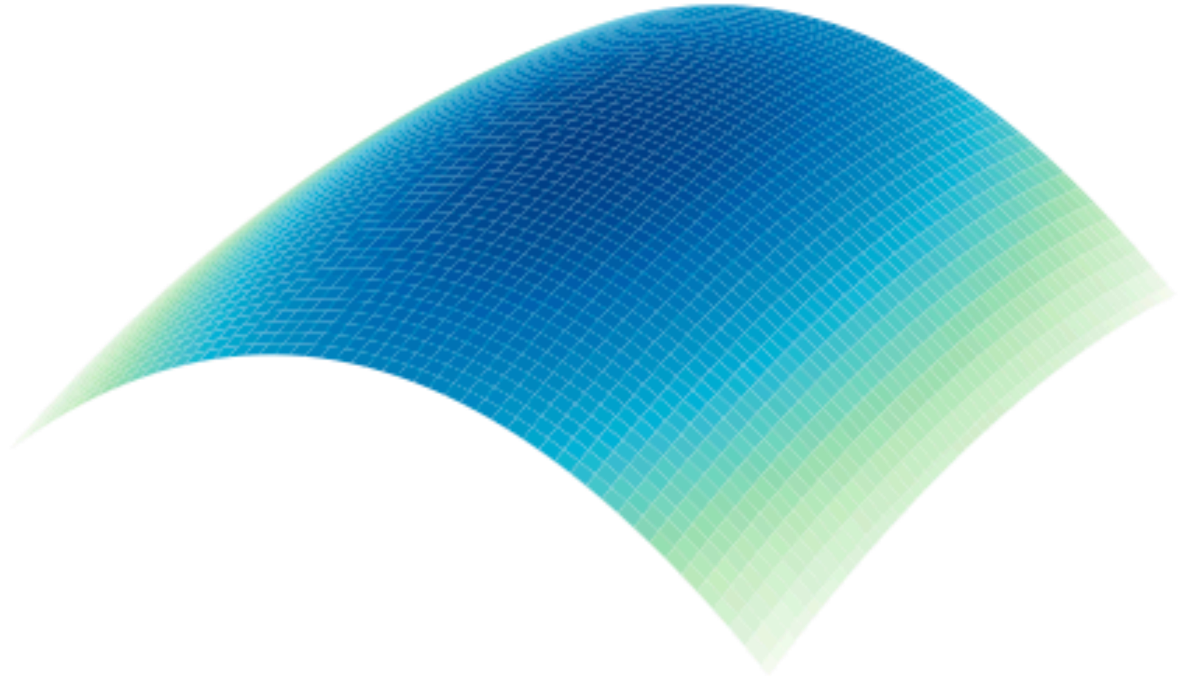


Manifolds and differential geometry

Manifold

generalization of the notion of a (n-dimensional) surface

set that can be parametrized by n numbers (coordinates) near any point



Manifolds and differential geometry

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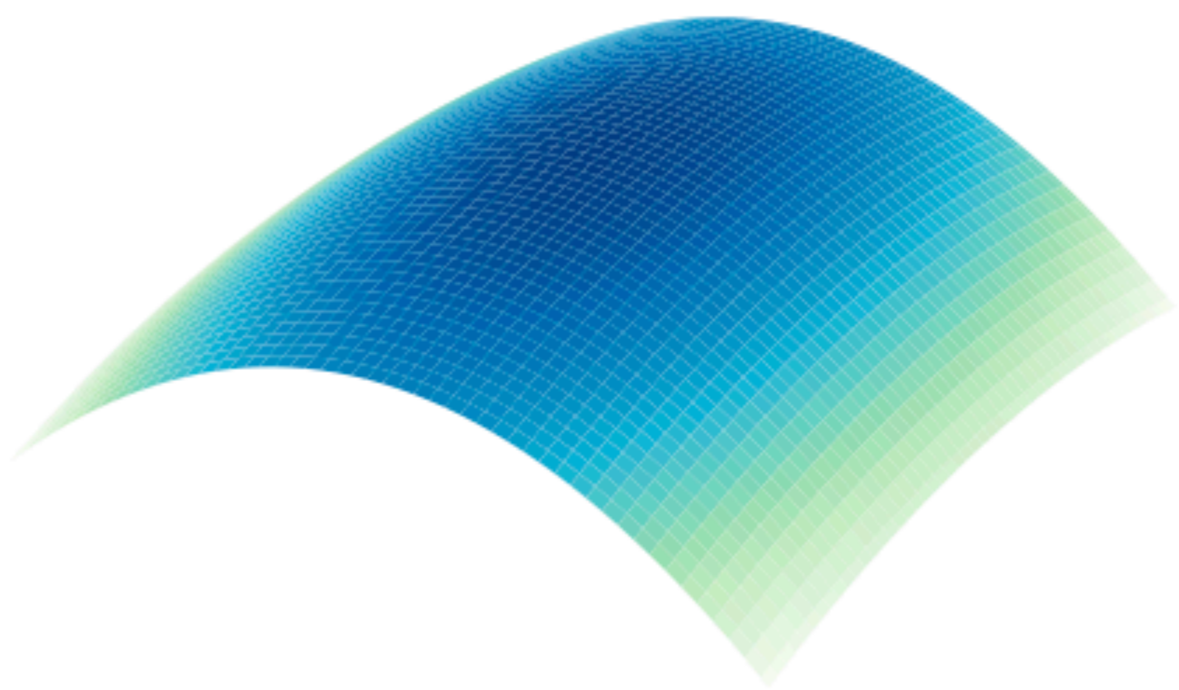
we can use the machinery of multivariate calculus (differentiation, integration) on a manifold



Manifolds and differential geometry

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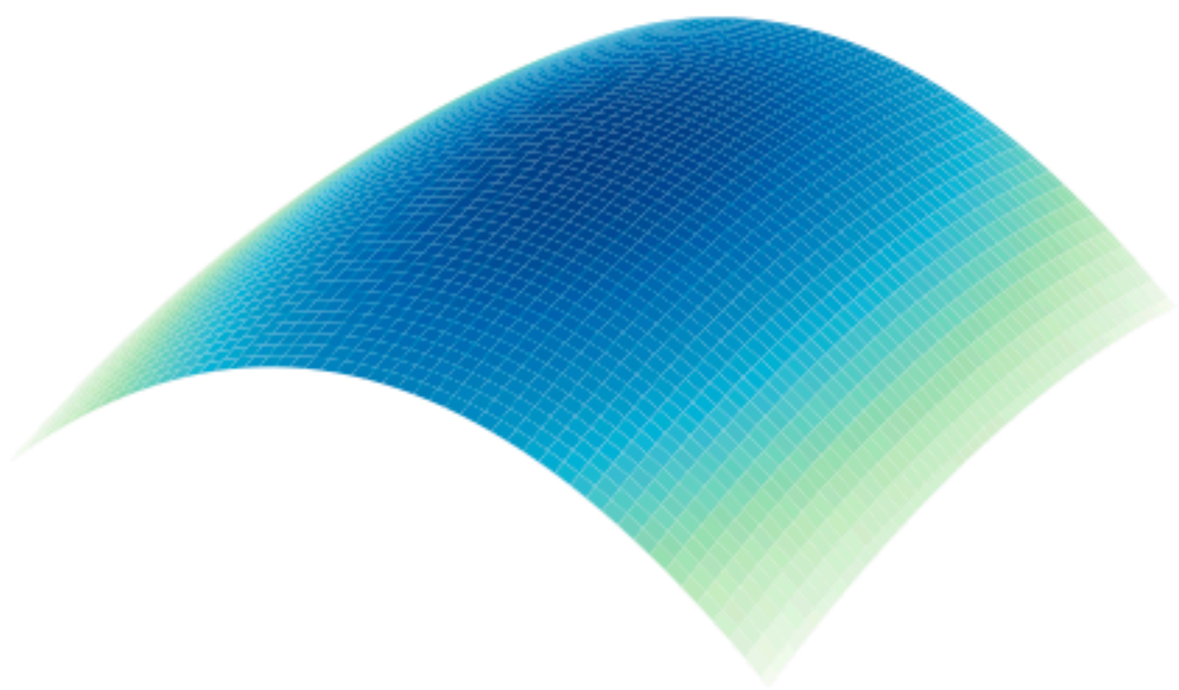

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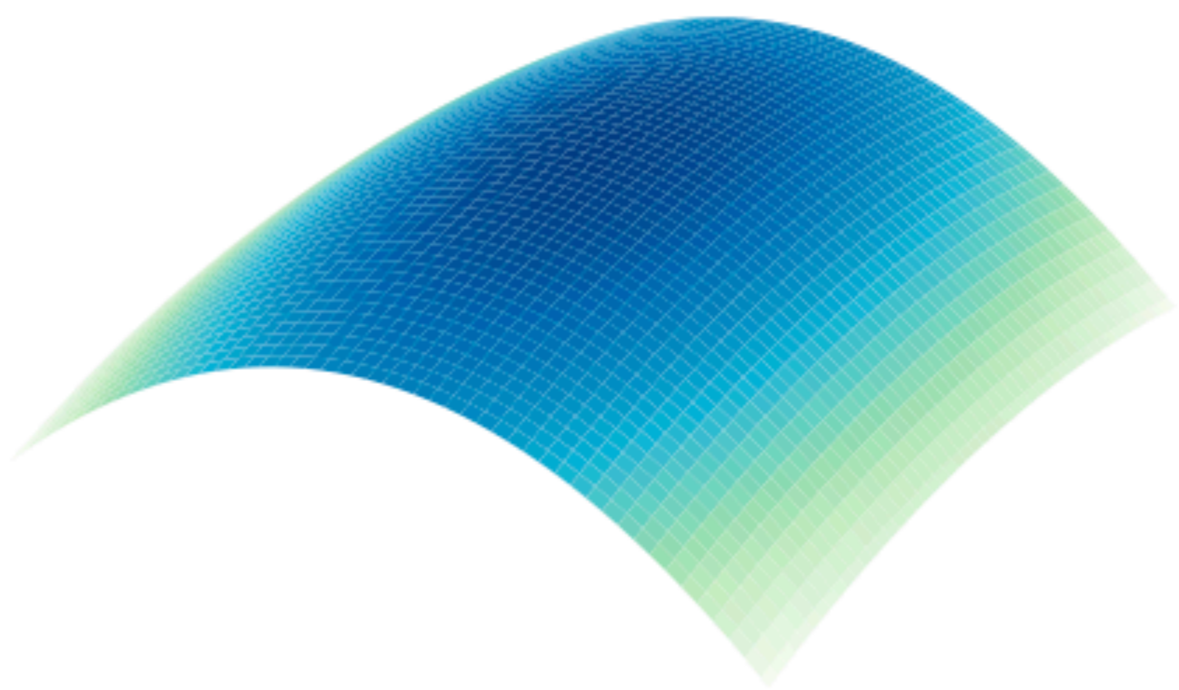
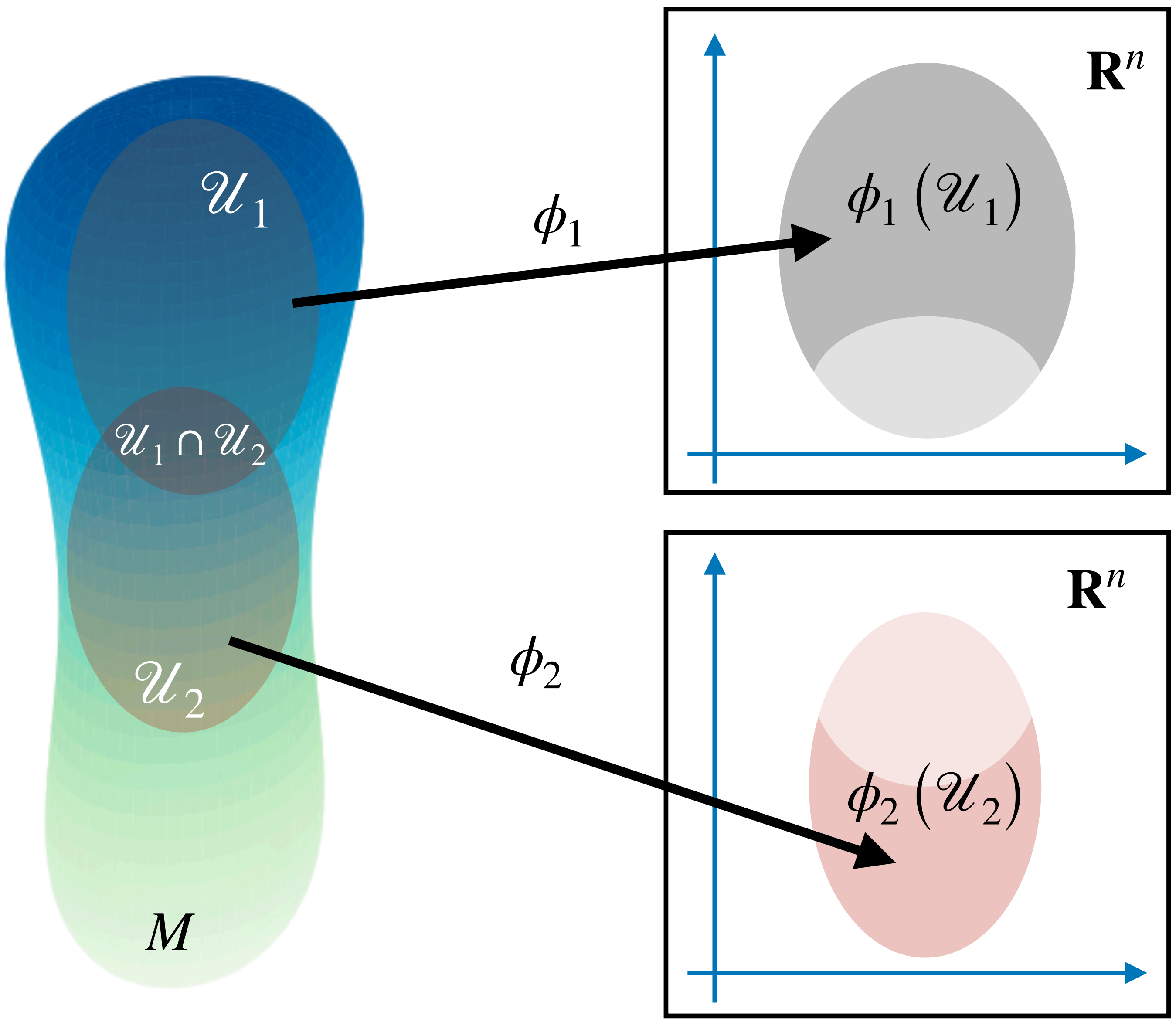


most important notion:
local coordinate system(s)

Manifolds and differential geometry

Manifold

$$M, \left\{ \left(\mathscr{U}_\alpha, \phi_\alpha \right) \right\}$$



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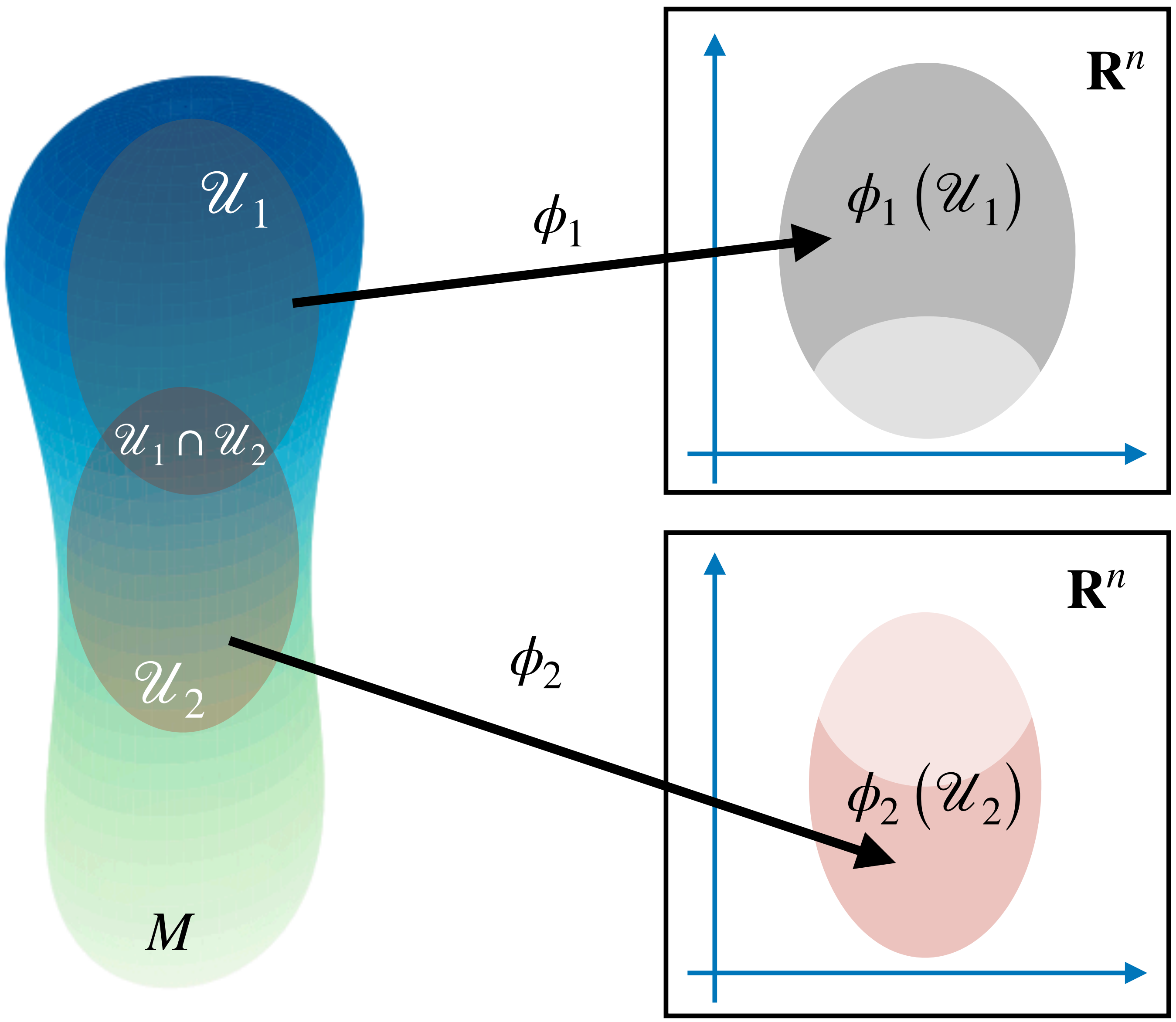
Manifolds and differential geometry

Manifold

$M, \left\{ \left(\mathcal{U}_\alpha, \phi_\alpha \right) \right\}$

topological space

atlas of charts (coordinate systems)



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Manifolds and differential geometry

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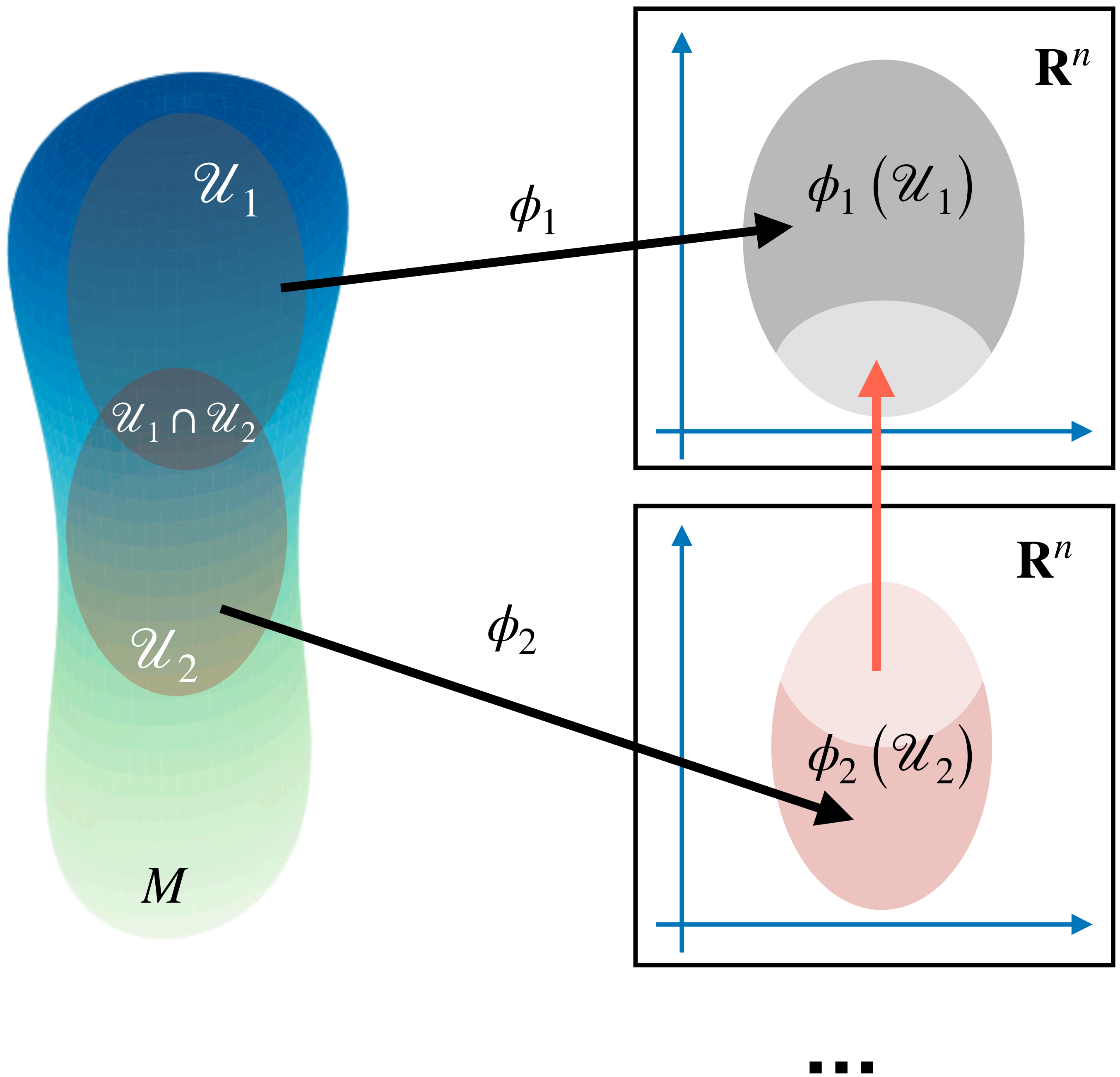

$$\bigcup_\alpha \mathscr{U}_\alpha = M$$

ϕ_α one-to-one

ϕ_α continuous

$\phi_\alpha \circ \phi_\beta^{-1}$ should be smooth (C^∞, continuously differentiable ∞ many times) on the overlap

$\Rightarrow \phi_\alpha \circ \phi_\beta^{-1}$ have non-singular Jacobian

$$\det\left(\frac{\partial x^i}{\partial y^{\bar{j}}}\right) \neq 0$$

Manifolds and differential geometry

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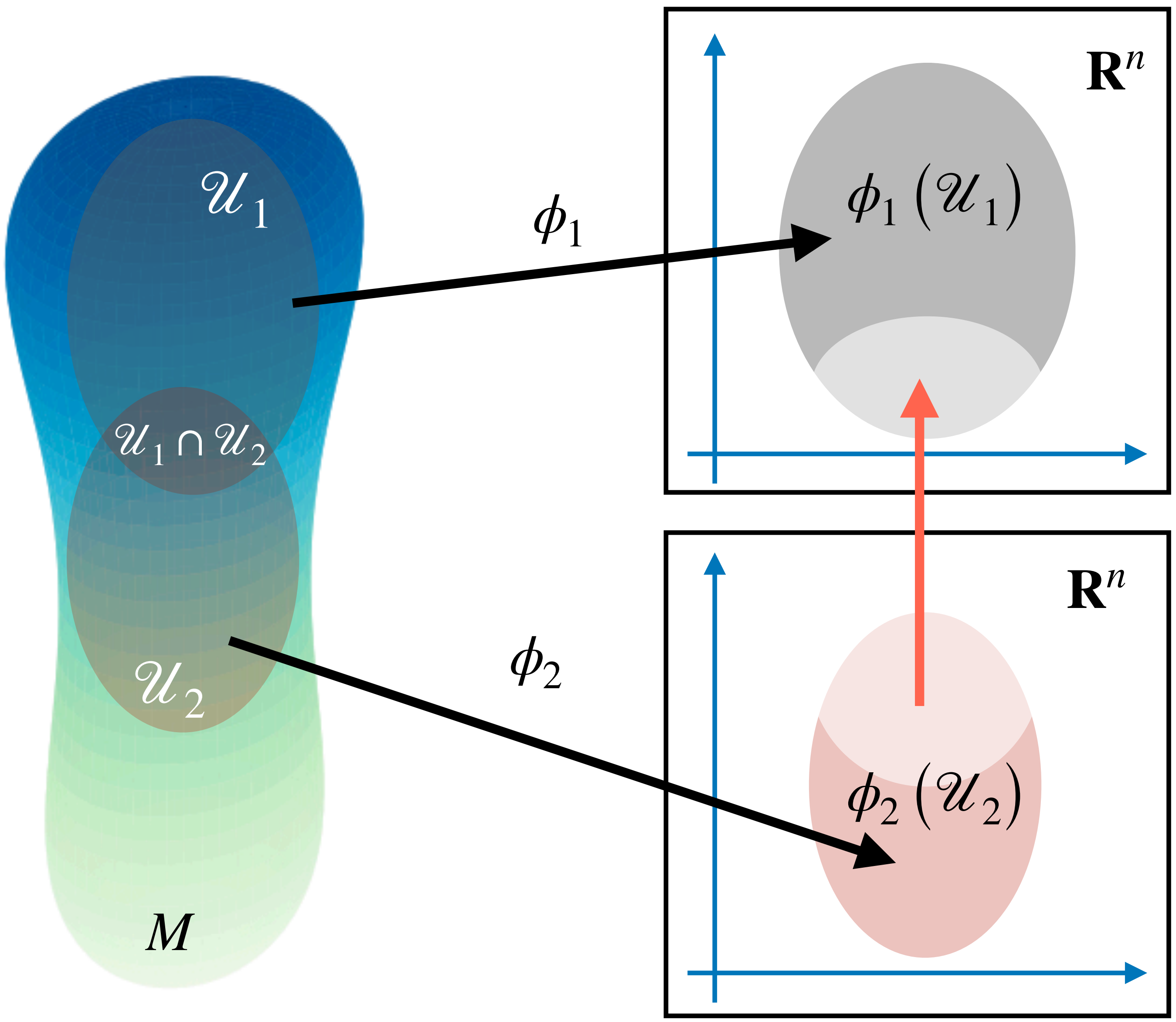

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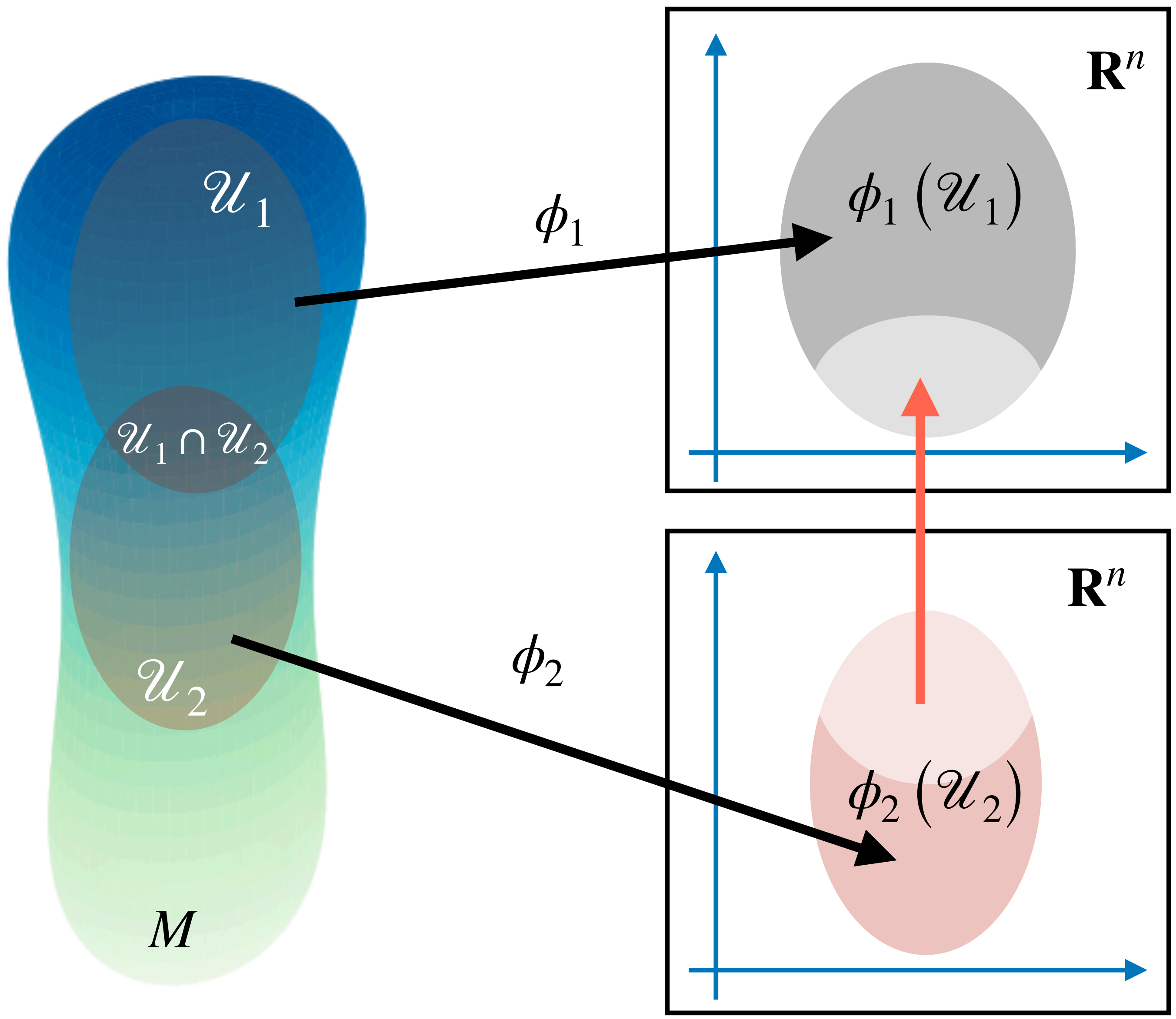

charts are usually local

...

Manifolds and differential geometry

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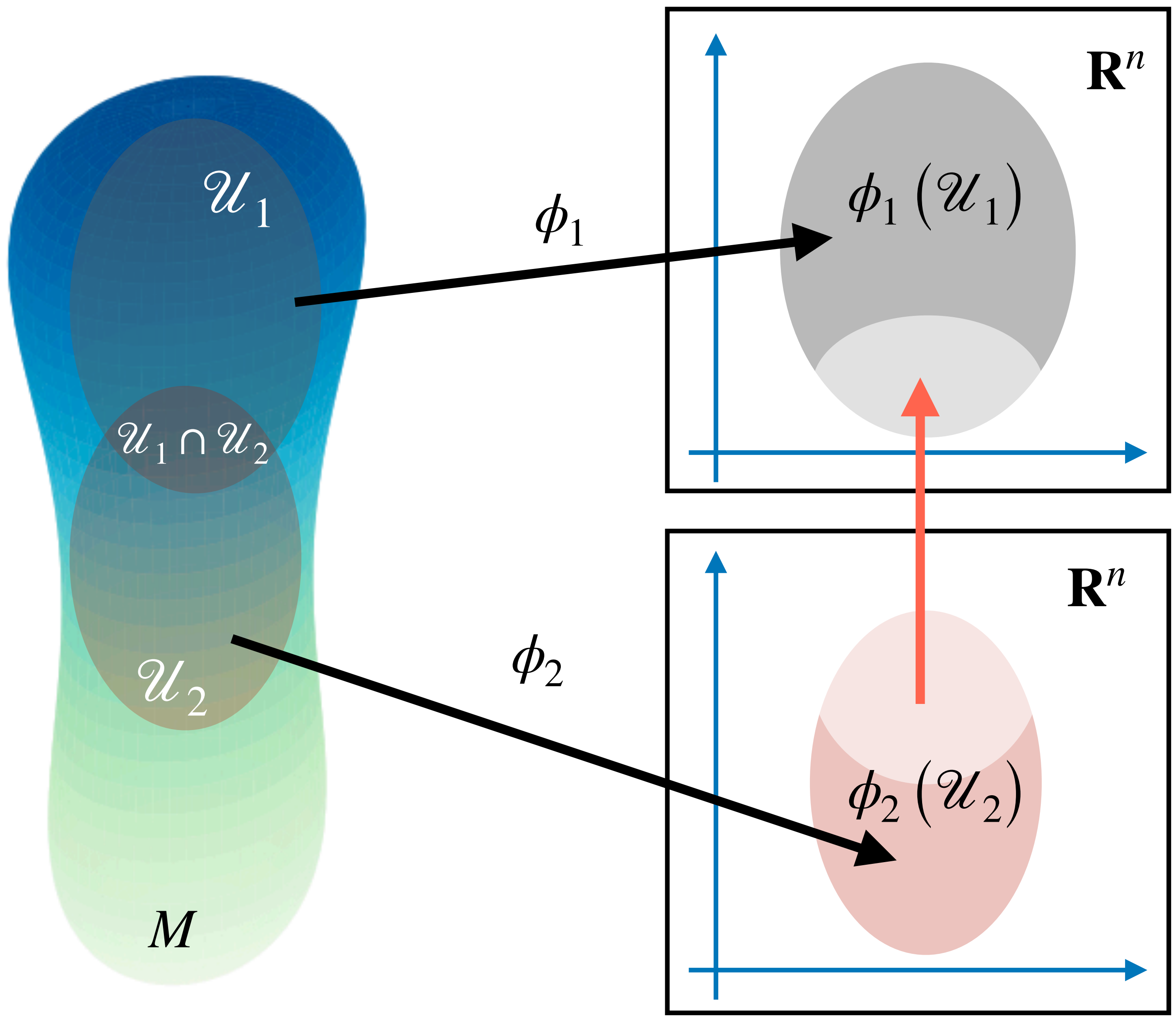

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many manifolds do not have a single, global chart

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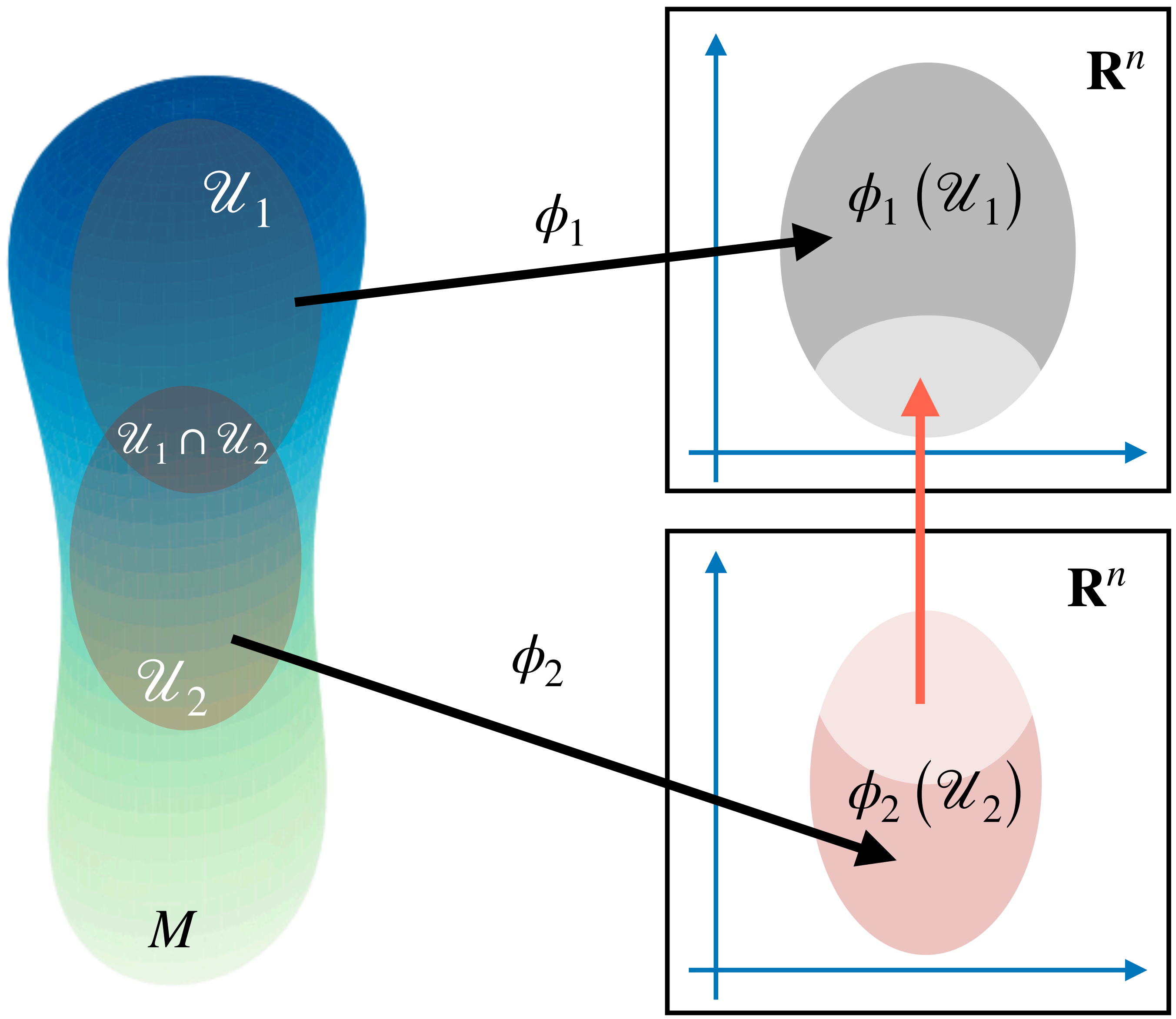

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transition maps
$(y^{\bar{1}}, y^{\bar{2}}, \ldots, y^{\bar{n}}) \mapsto (x^1, x^2, \ldots, x^n)$
coordinate transforms

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Manifolds and differential geometry

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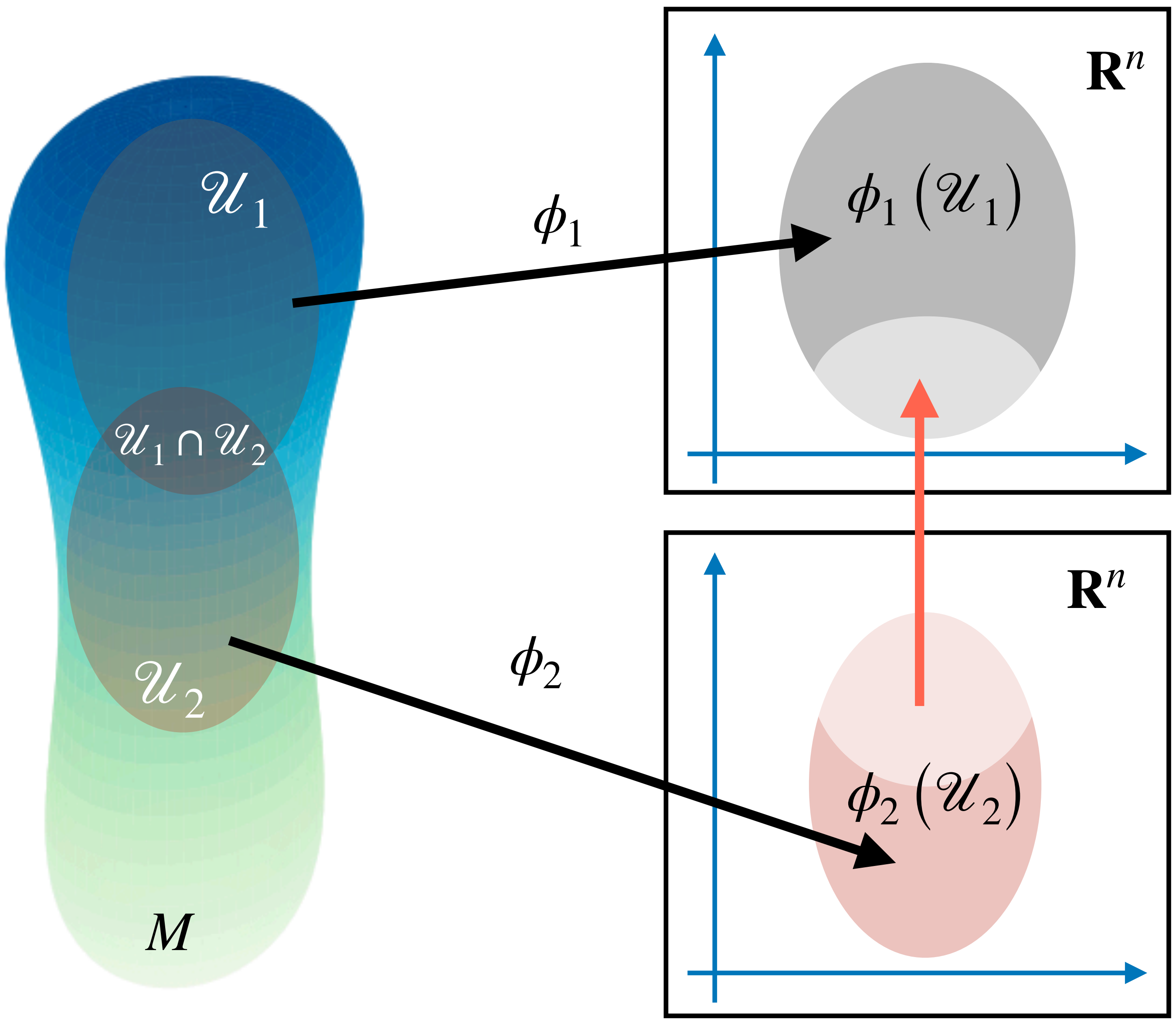

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coordinate transforms

We can always define new charts (enlarge the atlas), by picking the domain and transition maps

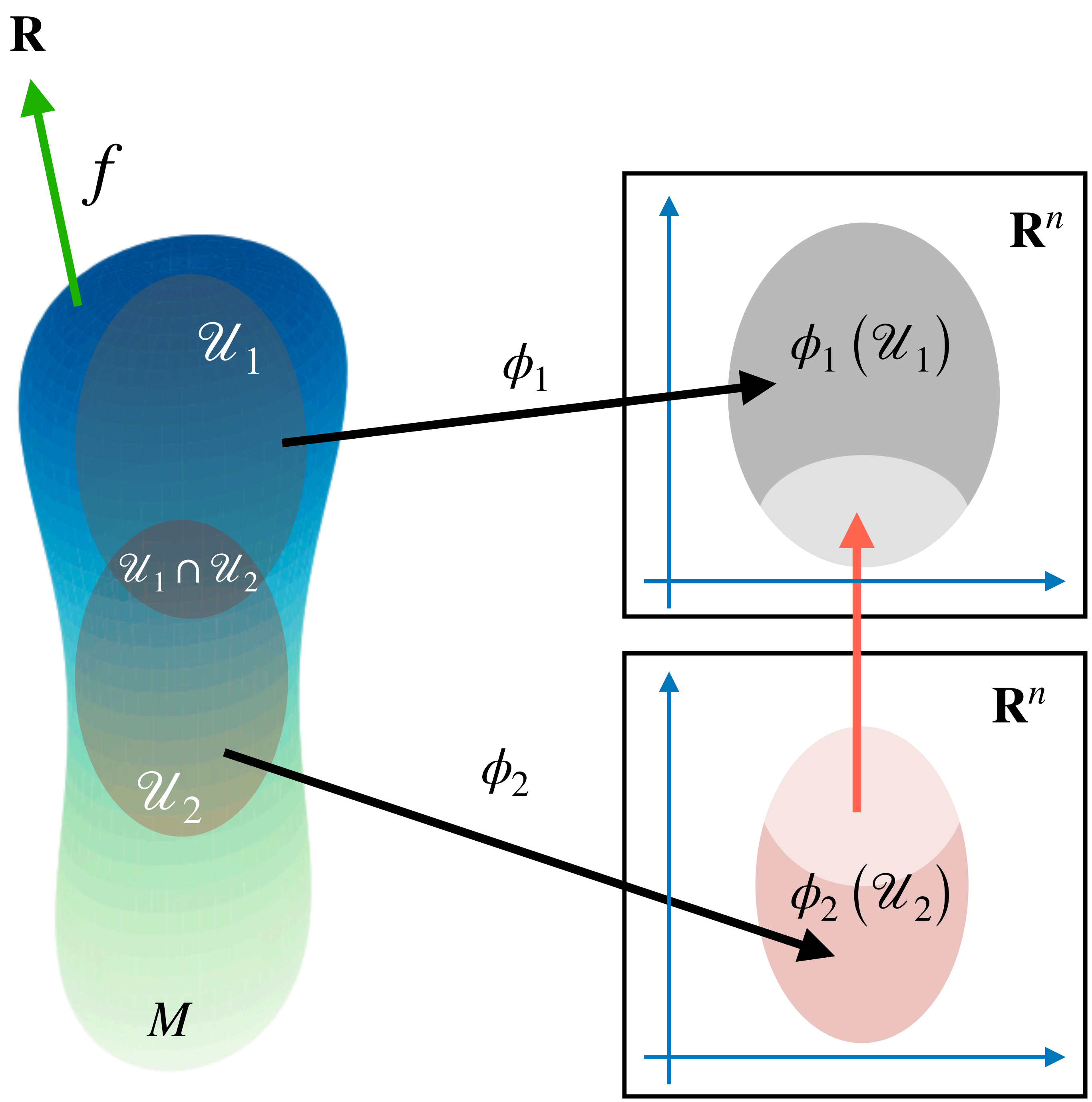
...

Manifolds and differential geometry

Functions (scalars)

coordinate systems

representation in coordinates



$f \circ \phi_1^{-1} : \phi_1(\mathcal{U}_1) \to \mathbf{R}$

$f(x^1, \ldots, x^n)$

$f \circ \phi_2^{-1} : \phi_2(\mathcal{U}_2) \to \mathbf{R}$

$f(y^{\bar{1}}, \ldots, y^{\bar{n}})$

...

Manifolds and differential geometry

Functions (scalars)

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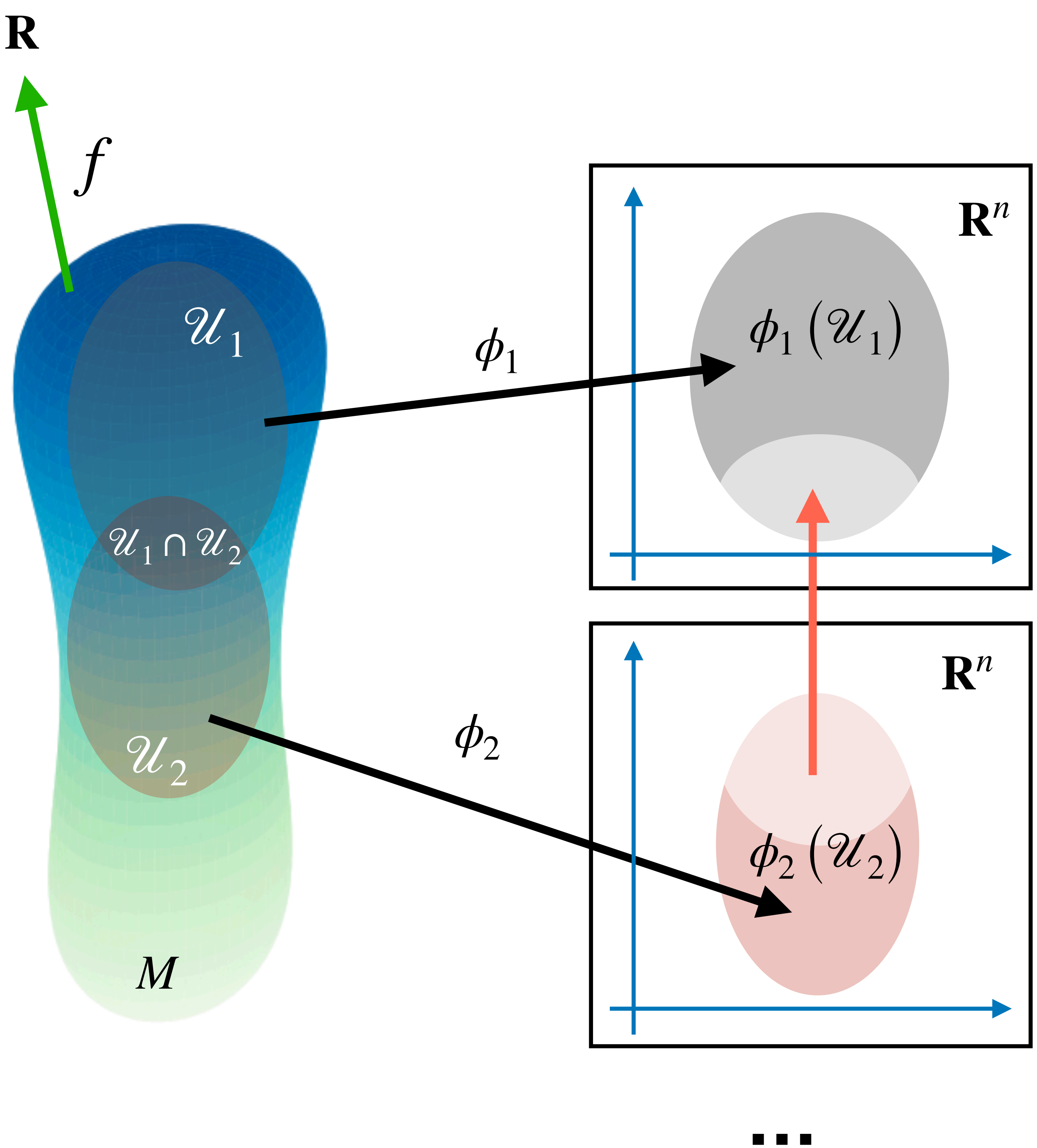

$$f \circ \phi_1^{-1}: \phi_1(\mathcal{U}_1) \to \mathbf{R}$$

$$f(x^1, \dots, x^n)$$

$$f\left(y^{\bar{k}}\right) = f\left(x^l\left(y^{\bar{k}}\right)\right)$$

$$f \circ \phi_2^{-1}: \phi_2(\mathcal{U}_2) \to \mathbf{R}$$

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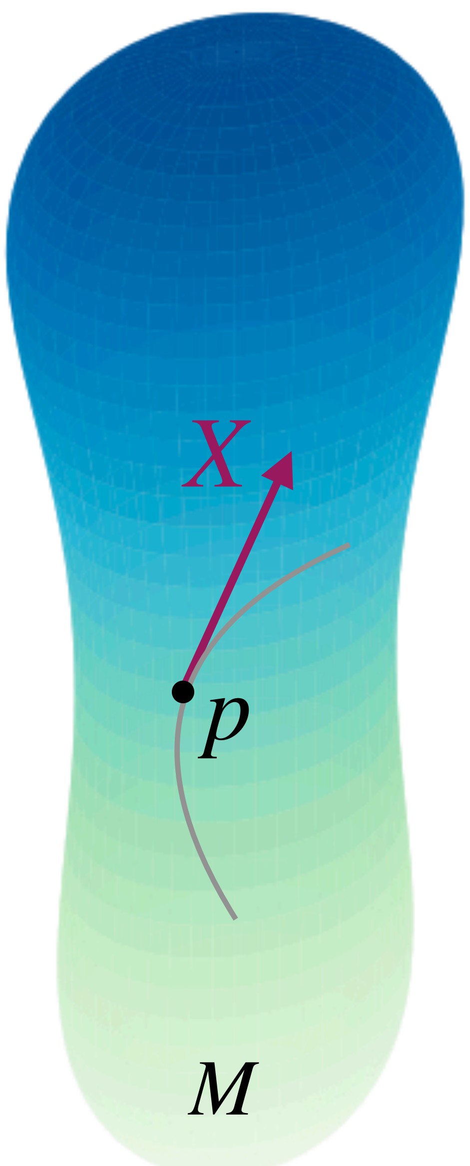
Manifolds and differential geometry

Vectors

vectors always defined *at a point*

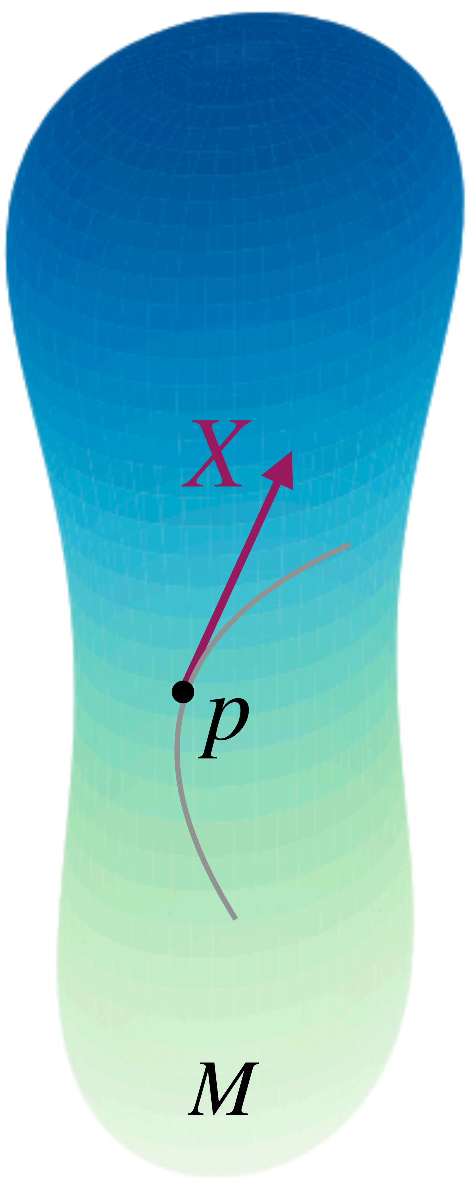


Manifolds and differential geometry

Vectors

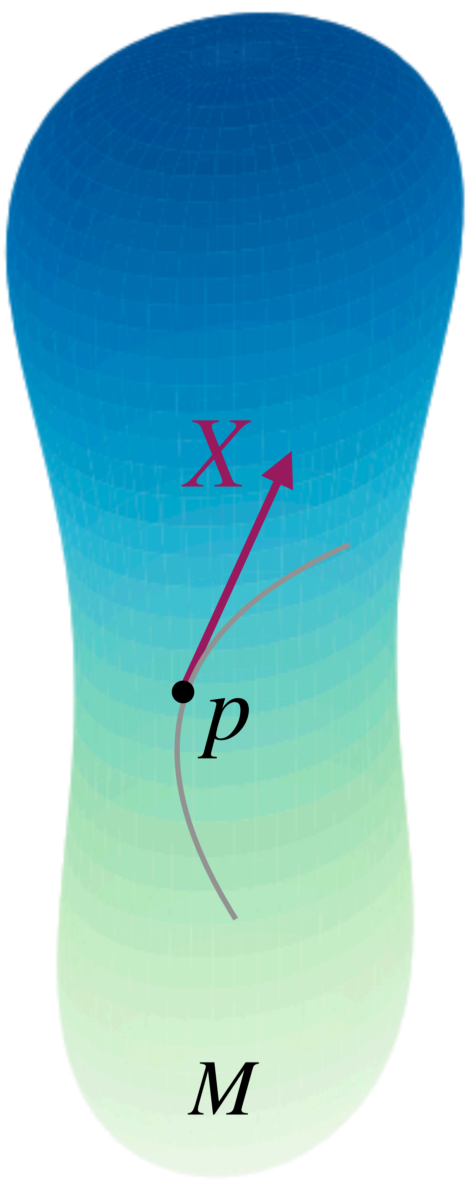
vectors always defined *at a point*

vectors at different points form different vector spaces (tangent spaces), we cannot add or combine them!



Manifolds and differential geometry

Vectors



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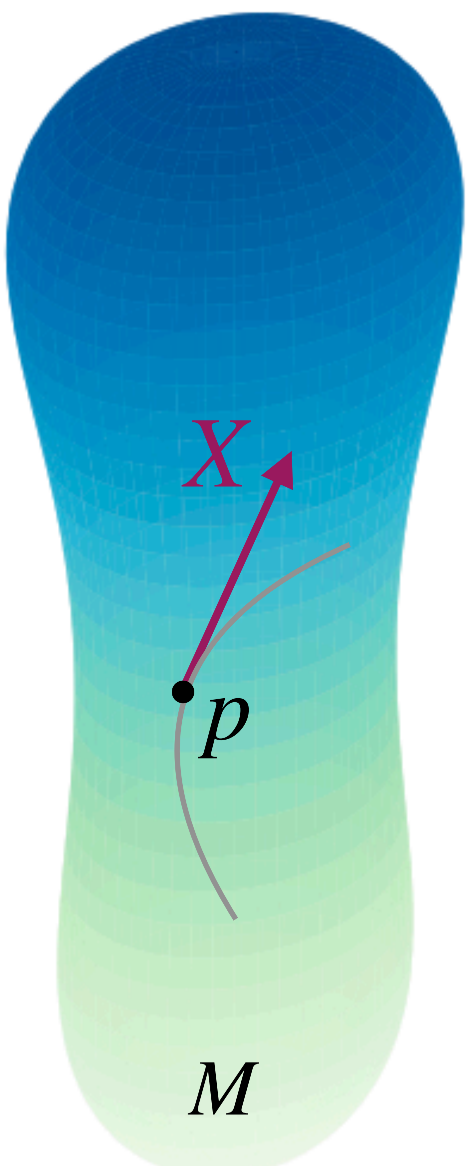
vectors do not connect distant points

Manifolds and differential geometry

Vectors



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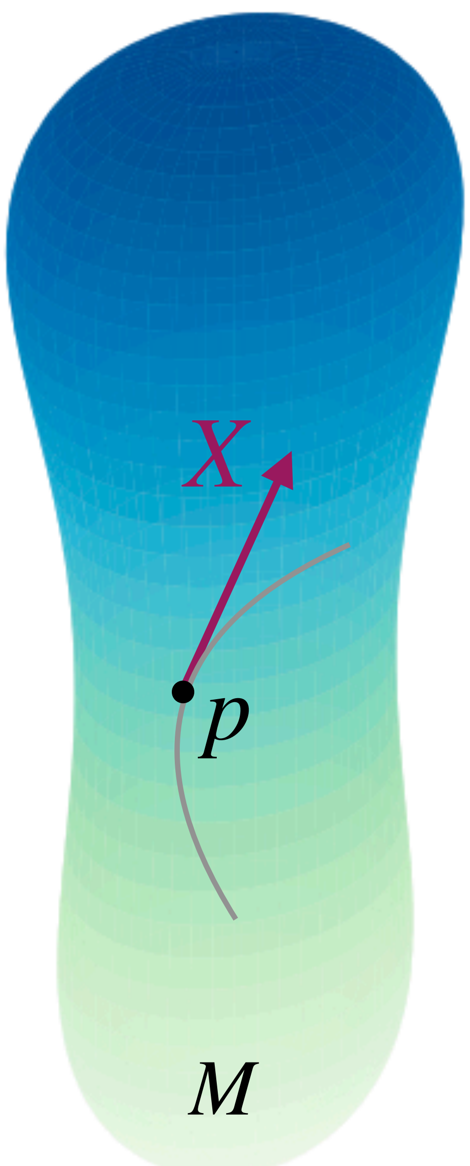
Manifolds and differential geometry

Vectors



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geometric intuitions

infinitesimal variation a point (or its coordinates)

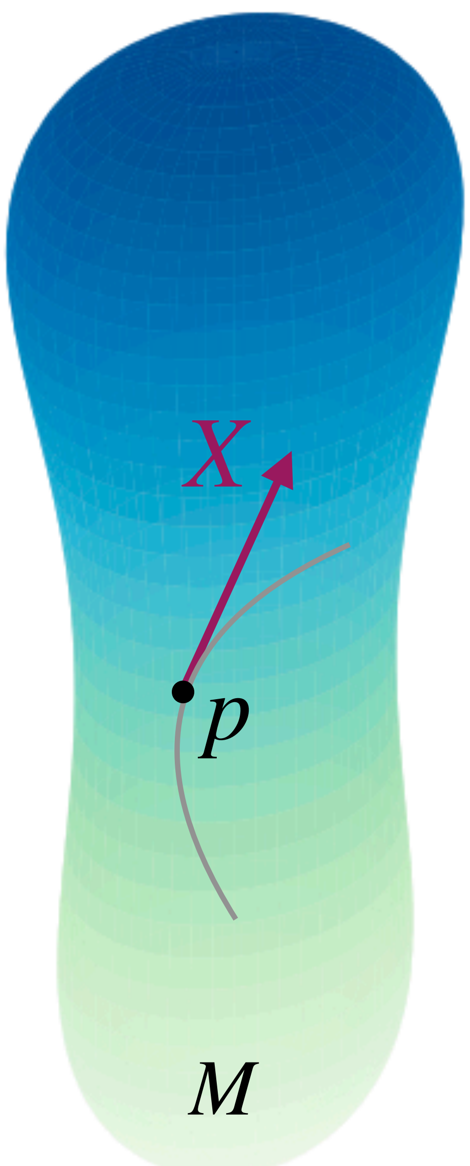
$$\delta x^i = X^i_p \, \delta\epsilon$$

tangent vector/velocity

$$X^i_p = \frac{dx^i}{d\lambda}$$

Manifolds and differential geometry

Vectors



transformation laws under coordinate transform

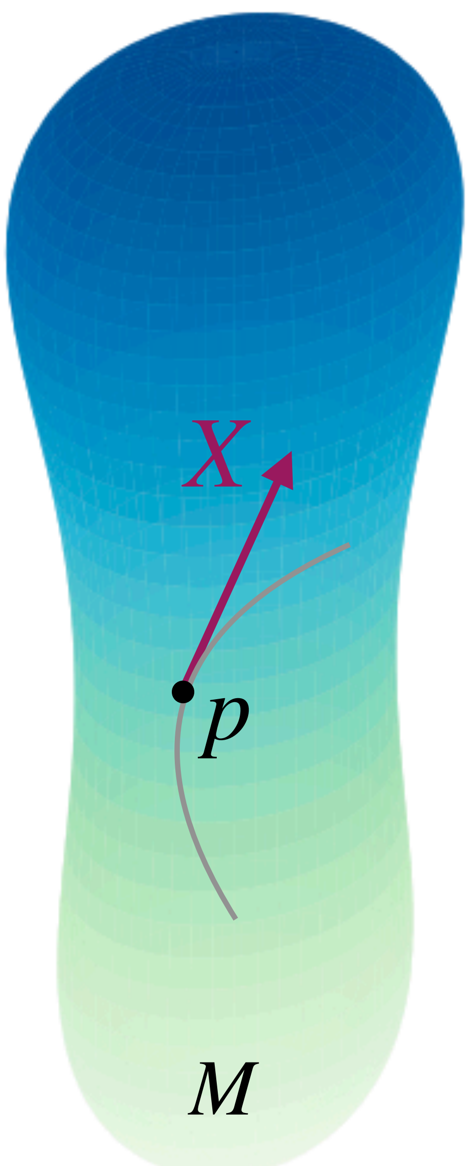
$$(y^{\bar{1}}, y^{\bar{2}}, \ldots, y^{\bar{n}}) \mapsto (x^1, x^2, \ldots, x^n)$$

$x^k\left(y^{\bar{i}}\right)$ given functions

$$X_p^{\bar{i}} = \frac{dy^{\bar{i}}}{d\lambda}$$

Manifolds and differential geometry

Vectors



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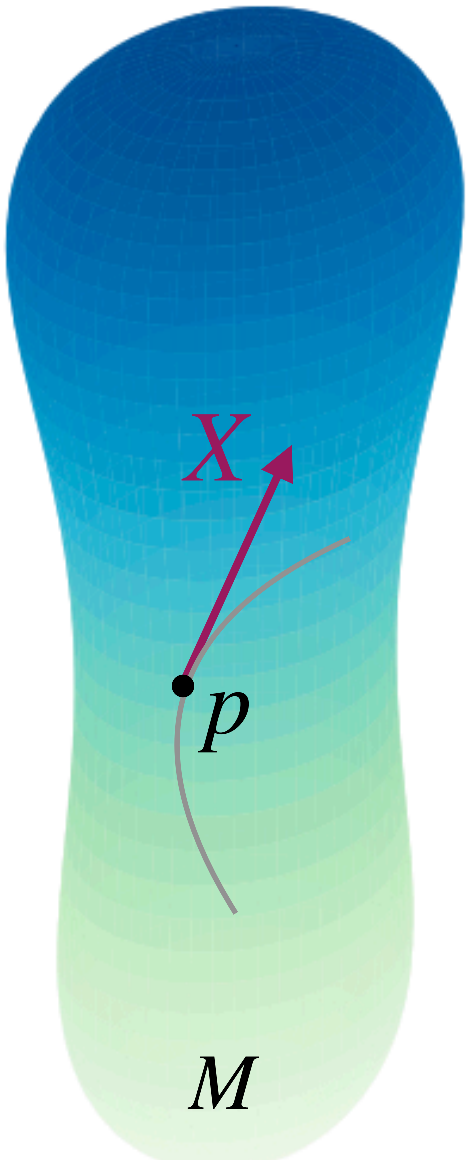
Manifolds and differential geometry

Vectors



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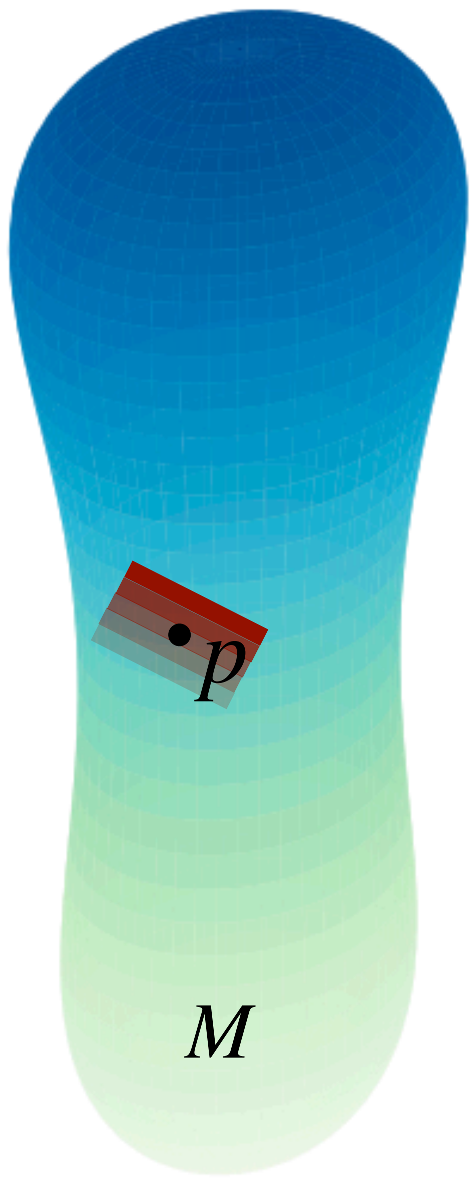
Jacobian $\left(\frac{\partial x}{\partial y}\right)$

Co-vectors

co-vectors also always defined *at a point*



Manifolds and differential geometry

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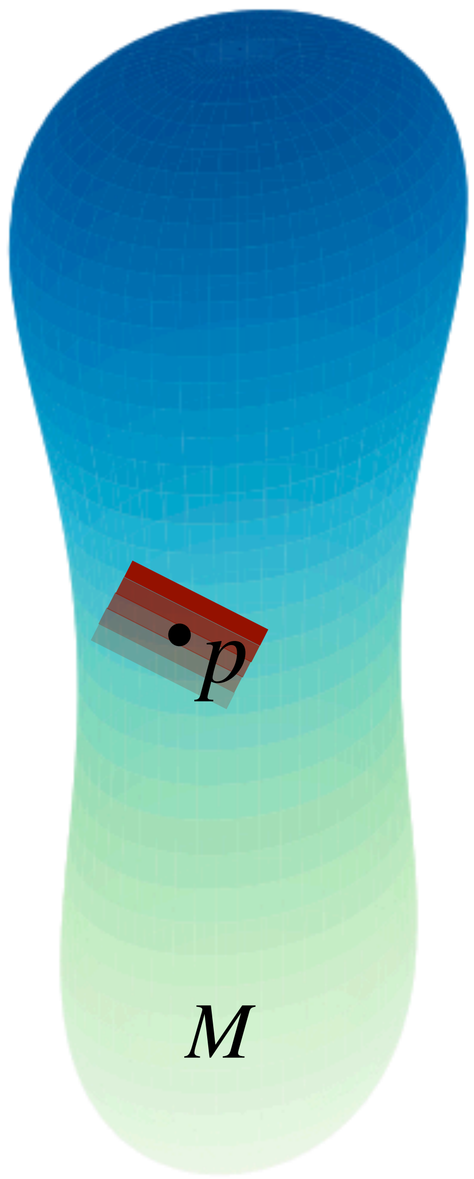
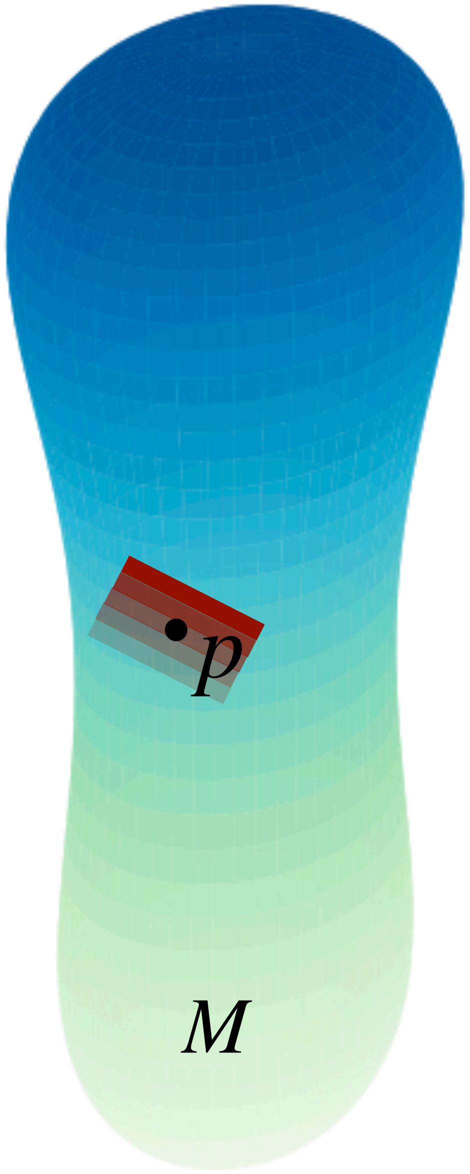


Manifolds and differential geometry

Co-vectors



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dimension of co-tangent spaces $\dim T_p^* M = n$

Manifolds and differential geometry

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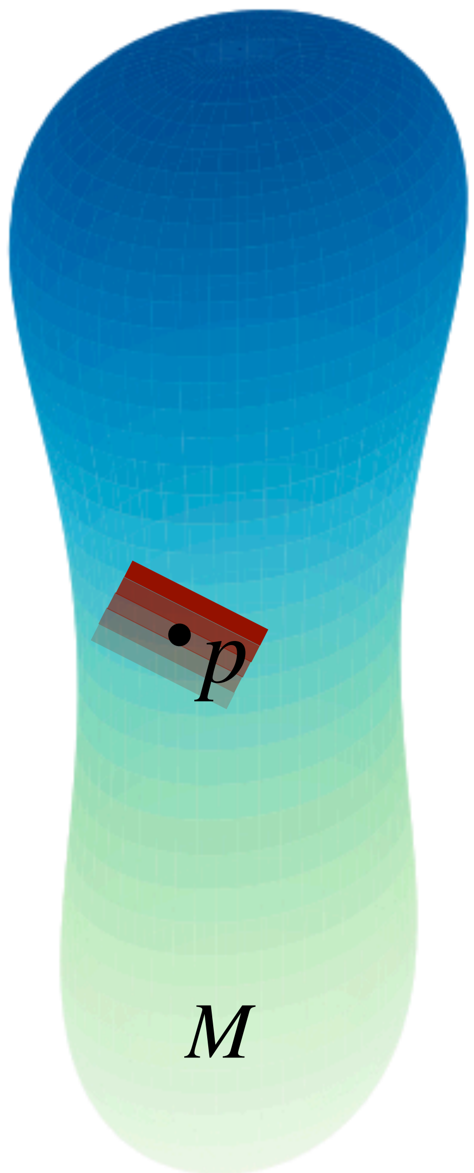


geometric intuitions

infinitesimal variation of a function

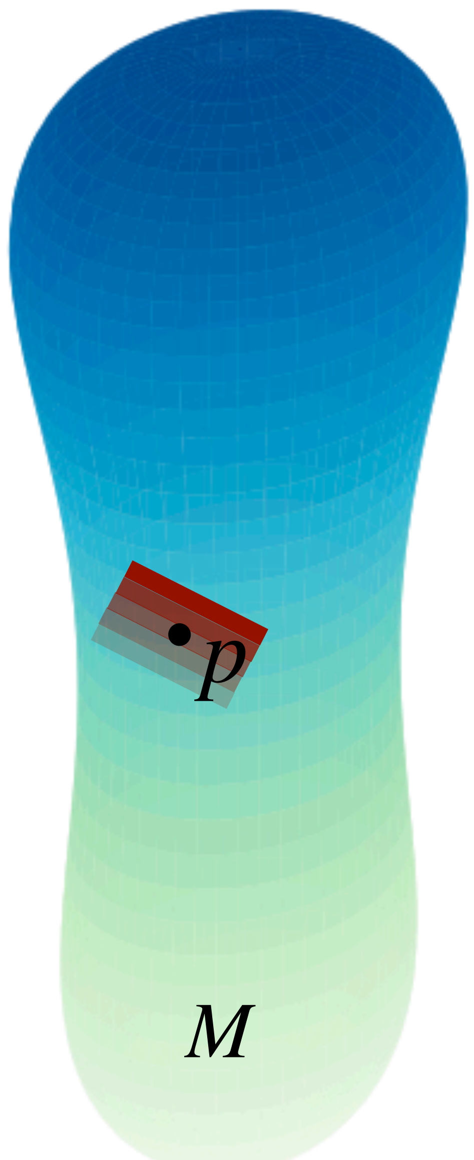
$$\delta f = \omega_i \, \delta x^i$$

gradient

$$\omega_i(p) = \left.\frac{\partial f}{\partial x^i}\right|_p$$

Manifolds and differential geometry

Co-vectors



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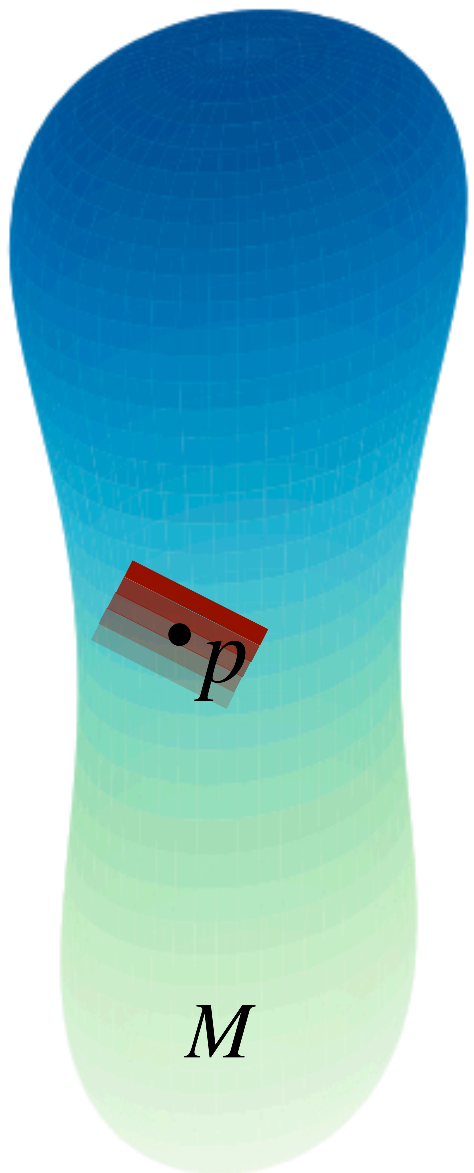
Manifolds and differential geometry

Co-vectors



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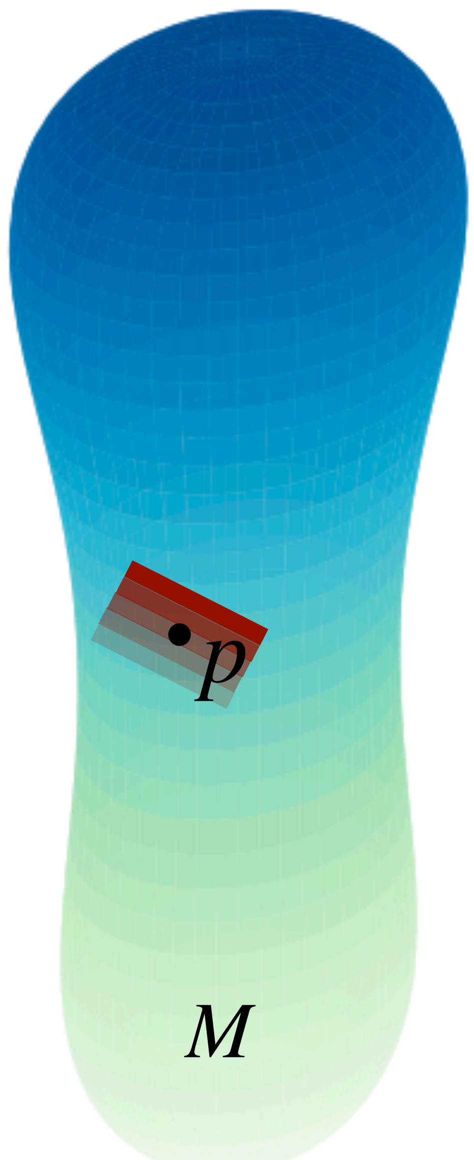
Manifolds and differential geometry

Co-vectors



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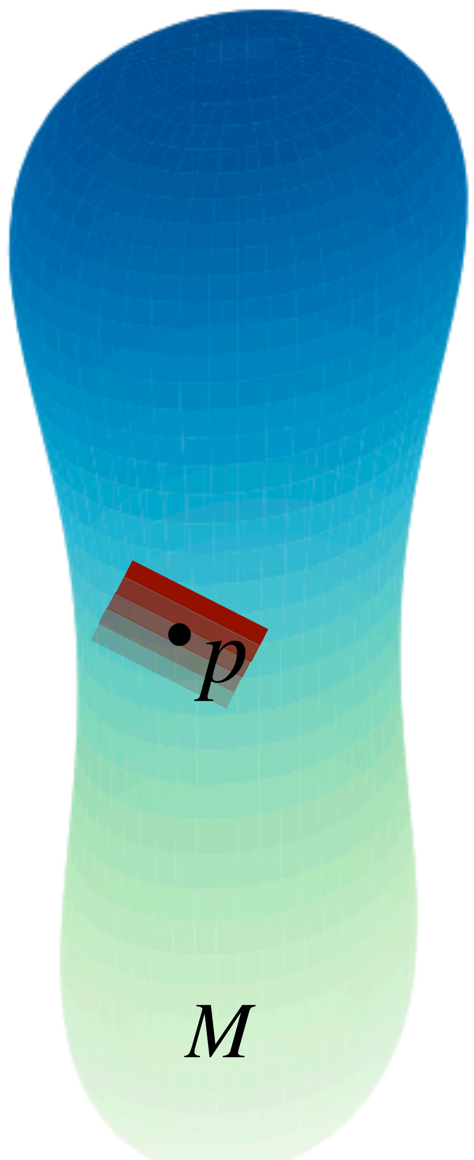
Manifolds and differential geometry

Co-vectors



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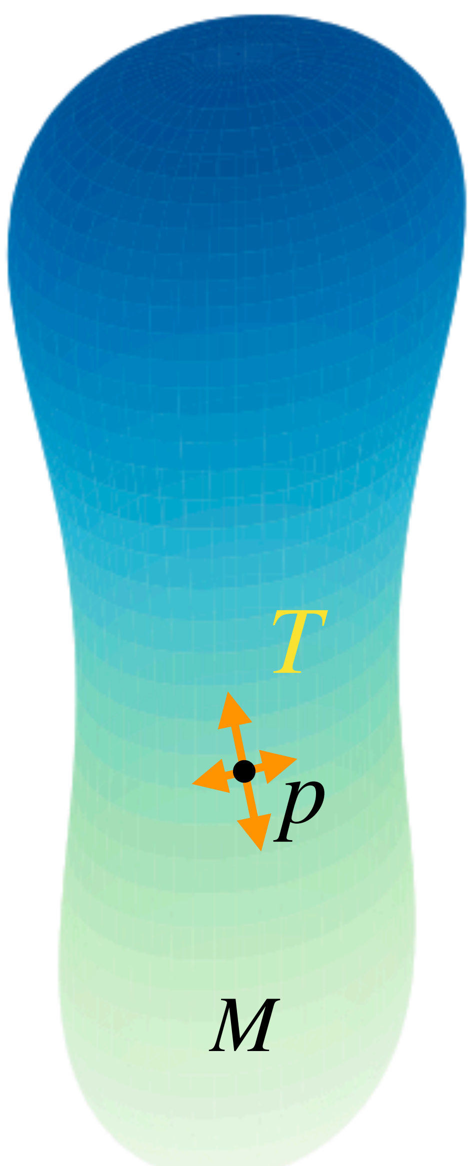
$$\Rightarrow \omega_{\bar{i}}(p) \rightarrow \omega_i(p) = \omega_{\bar{i}}(p) \cdot \frac{\partial y^{\bar{i}}}{\partial x^i}\bigg|_p$$

$$\frac{\partial x^i}{\partial y^{\bar{j}}} \cdot \frac{\partial y^{\bar{j}}}{\partial x^k} = \delta^i{}_k$$

inverse Jacobian $\left(\frac{\partial y}{\partial x}\right) = \left(\frac{\partial x}{\partial y}\right)^{-1}$

Manifolds and differential geometry

Tensors



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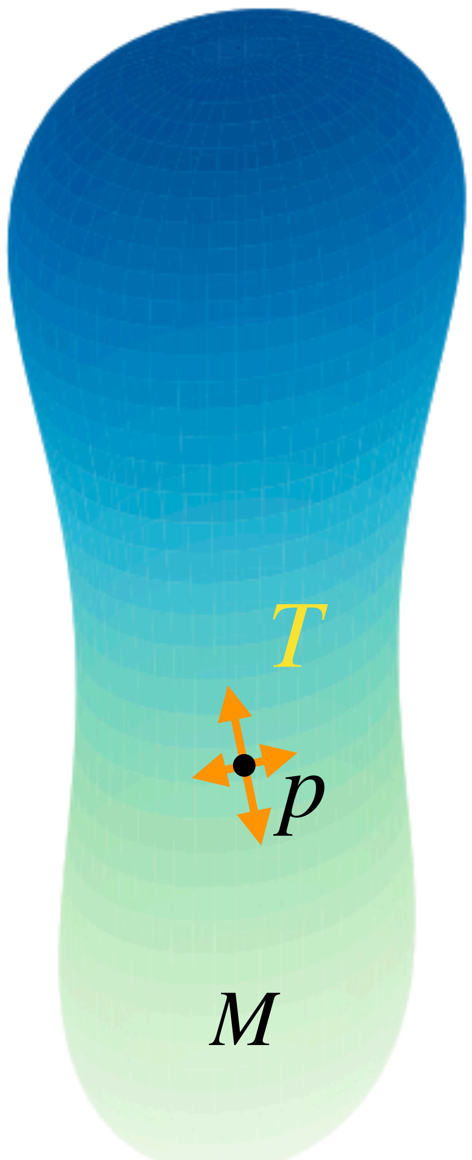
$x^k\left(y^{\bar{i}}\right)$ given functions

$T^{\bar{i}\bar{j}\ldots}{}_{\bar{k}\bar{l}\ldots}(p)$ tensor coordinates in $\left(y^{\bar{i}}\right)$

Manifolds and differential geometry

Tensors



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inverse Jacobian $\left(\frac{\partial y}{\partial x}\right) = \left(\frac{\partial x}{\partial y}\right)^{-1}$

$$T^{\bar{i}\bar{j}\ldots}{}_{\bar{k}\bar{l}\ldots}(p) \to T^{ij\ldots}{}_{kl\ldots}(p) = T^{\bar{i}\bar{j}\ldots}{}_{\bar{k}\bar{l}\ldots}(p) \cdot \left.\frac{\partial x^i}{\partial y^{\bar{i}}}\right|_p \left.\frac{\partial x^j}{\partial y^{\bar{j}}}\right|_p \cdots \left.\frac{\partial y^{\bar{k}}}{\partial x^k}\right|_p \left.\frac{\partial y^{\bar{l}}}{\partial x^l}\right|_p \cdots$$

Manifolds and differential geometry

Remarks

We usually work with vector/co-vector/tensor ***fields*** $T^{ij\ldots}{}_{kl\ldots}\left(x^m\right)$

Need to change the argument when changing coordinates

Manifolds and differential geometry

Remarks

We usually work with vector/co-vector/tensor ***fields*** $T^{ij\dots}{}_{kl\dots}\left(x^m\right)$

Need to change the argument when changing coordinates

At each (co-)tangent space we are decomposing tensors in so-called coordinate basis, related to a given coordinate system (say $\left(x^i\right)$). **The basis usually isn't orthonormal.**

$$X_p = X_p^i\, e_i \equiv X_p^i\, \partial_i \equiv X_p^i \left.\frac{\partial}{\partial x^i}\right|_p$$

$$\kappa(p) = \kappa_i(p)\, \omega^i \equiv \kappa_i(p)\, \mathrm{d}x^i$$

Manifolds and differential geometry

Remarks

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It is possible to use bases unrelated to the current coordinate system, or bases unrelated to ***any*** coordinate system (non-coordinate bases)

Manifolds and differential geometry

Remarks

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It is possible to use bases unrelated to the current coordinate system, or bases unrelated to ***any*** coordinate system (non-coordinate bases)

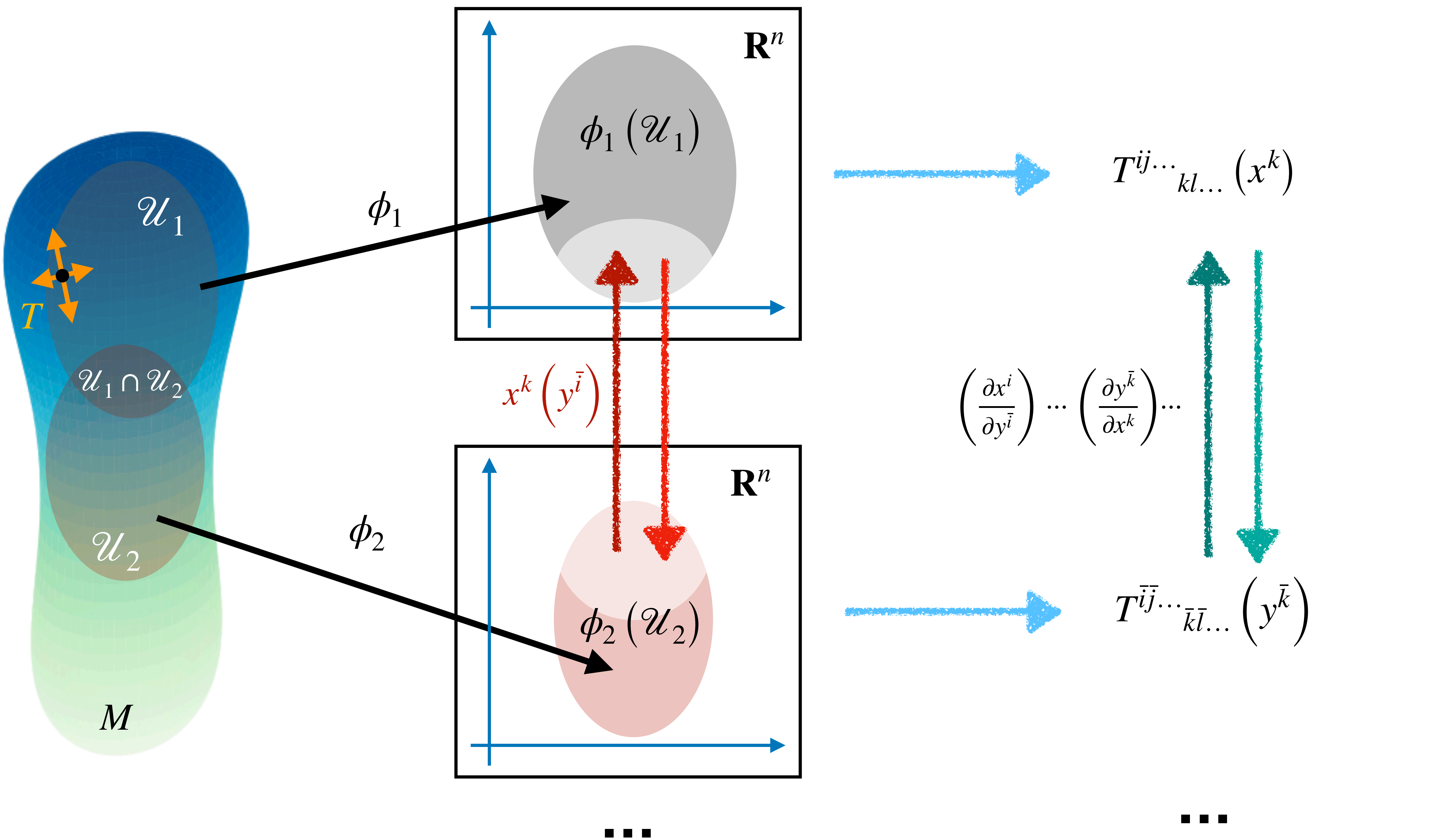
Coordinate transformations = very important thing in GR, we will practice that.

Manifolds and differential geometry

Manifold + tensor field

coordinate systems
(coordinate bases)

representation in coordinates
(in coordinate basis)



End of Lecture 3