Tensor algebra

Tensors

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Describe physical quantities of more complicated geometric character

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Known in non-relativistic physics (continuous media)

Moment of inertia tensor

Stress tensor

Permittivity tensor

…

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Used extensively in quantum information theory

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Used extensively in quantum information theory

Fundamental objects in GR and SR

- most important fields/quantities are tensorial
- tensors combine various 3D quantities into a single 4D object

Tensor algebra

Geometric objects are defined by their transformation laws under the change of frames

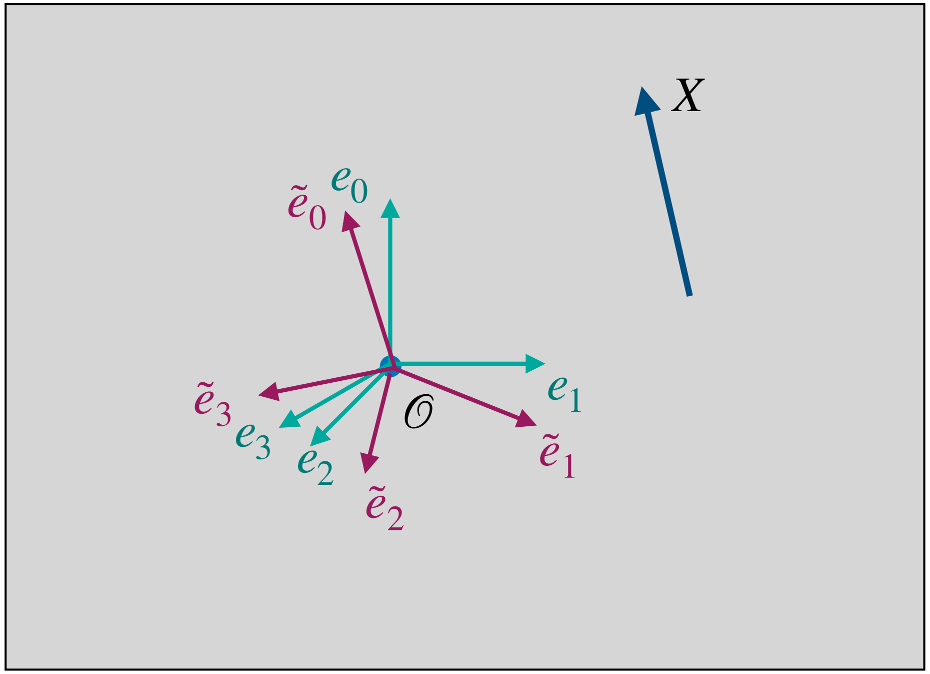
Scalars

Number defined at a point

Frame-independent

$$\{e_\mu\} \to \{\tilde{e}_\mu\}, \quad \tilde{e}_\mu = (\Lambda^{-1})^\nu{}_\mu \, e_\nu$$

$$s \to \tilde{s} = s$$



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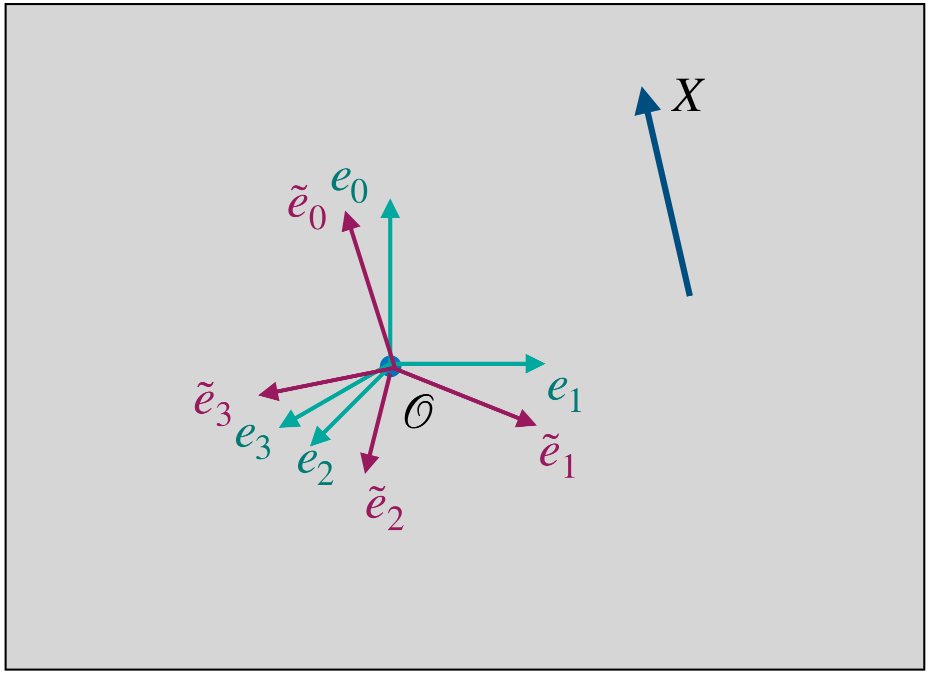


Vectors

$$X = X^\mu \, e_\mu$$

$$X = \tilde{X}^\mu \, \tilde{e}_\mu = \underbrace{\tilde{X}^\mu \, (\Lambda^{-1})^\sigma{}_\mu}_{X^\sigma} \, e_\sigma$$

$$\{e_\mu\} \to \{\tilde{e}_\mu\}, \quad \tilde{e}_\mu = (\Lambda^{-1})^\nu{}_\mu \, e_\nu$$

$$X^\mu \to \tilde{X}^\mu = \Lambda^\mu{}_\nu \, X^\nu$$

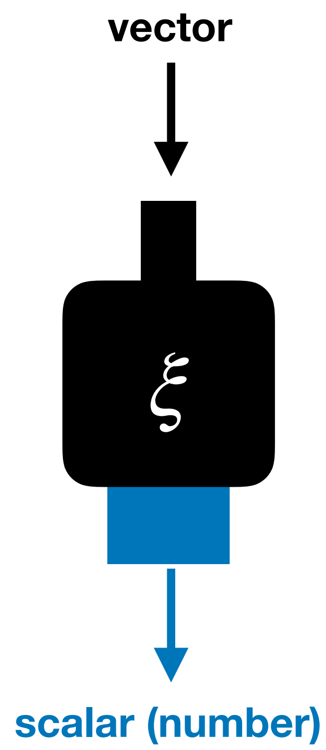
Intuition: displacements, velocities…

Tensor algebra

Co-vectors / one-forms

Co-vector = linear function of vectors, yielding scalars

$$\xi(X) = \xi_\mu X^\mu \qquad \xi_\mu := \xi(e_\mu)$$



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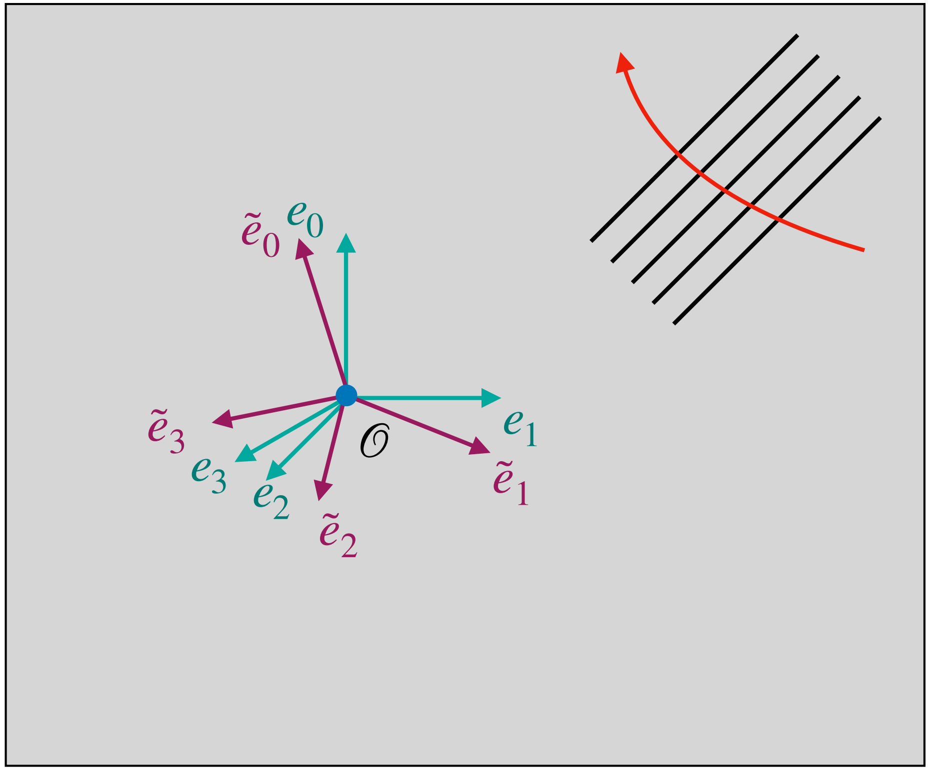
$$\xi_\nu \to \tilde{\xi}_\nu = (\Lambda^{-1})^\mu{}_\nu \, \xi_\mu$$



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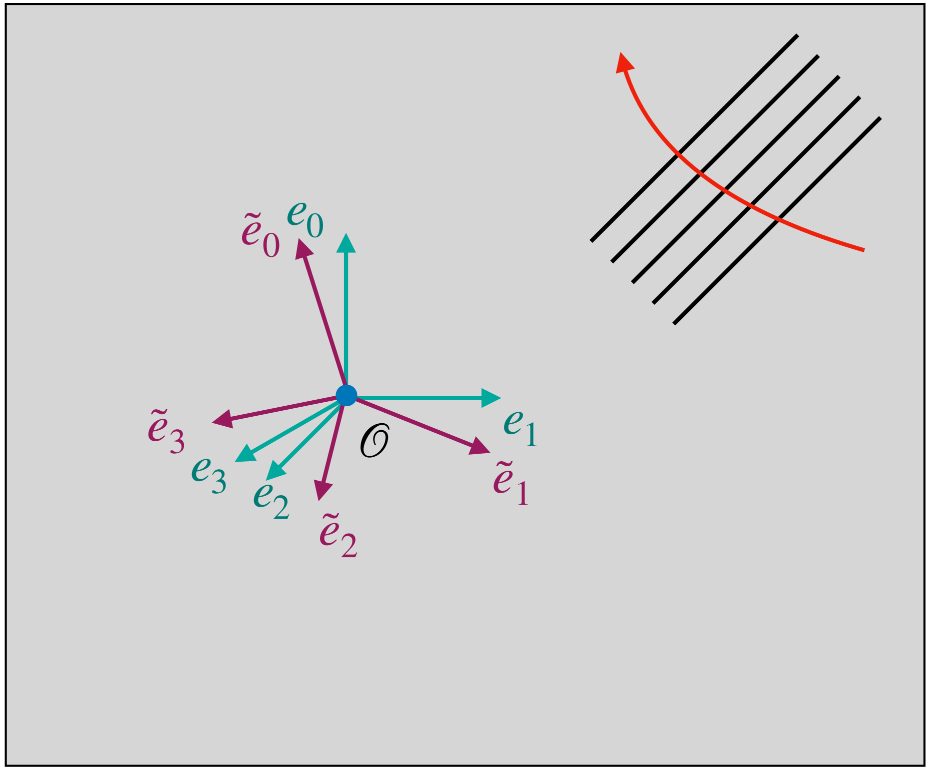
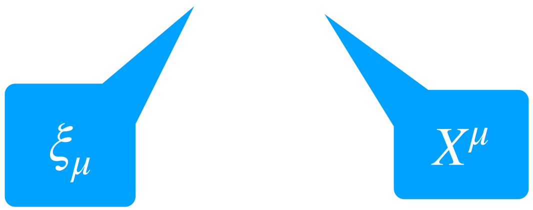


Intuition: gradient of a function

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^\mu} \frac{dx^\mu}{d\lambda}$$

function $f(x^\mu)$

curve $x^\mu(\lambda)$



Tensor algebra

Co-vectors / one-forms

Dual basis of co-vectors

$\{e_\mu\}$ - basis of vectors

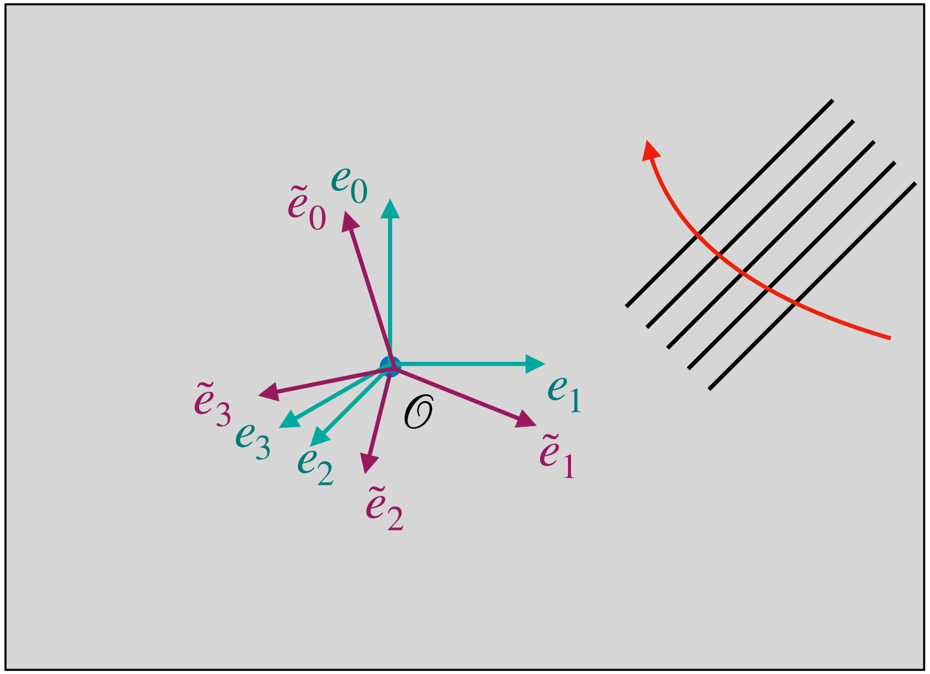
define $\omega^0, \omega^1, \omega^2, \omega^3$:

basis index, not component!

$$\omega^\alpha(e_\beta) = \delta^\alpha{}_\beta = \begin{cases} 1 & \text{if} \quad \alpha = \beta \\ 0 & \text{if} \quad \alpha \neq \beta \end{cases}$$

$\{\omega^\mu\}$ - basis of co-vectors, dual to $\{e_\mu\}$

$$\xi = \xi_\mu \, \omega^\mu \qquad \Longrightarrow \xi(X) = \xi_\mu \, X^\mu$$



Tensor algebra

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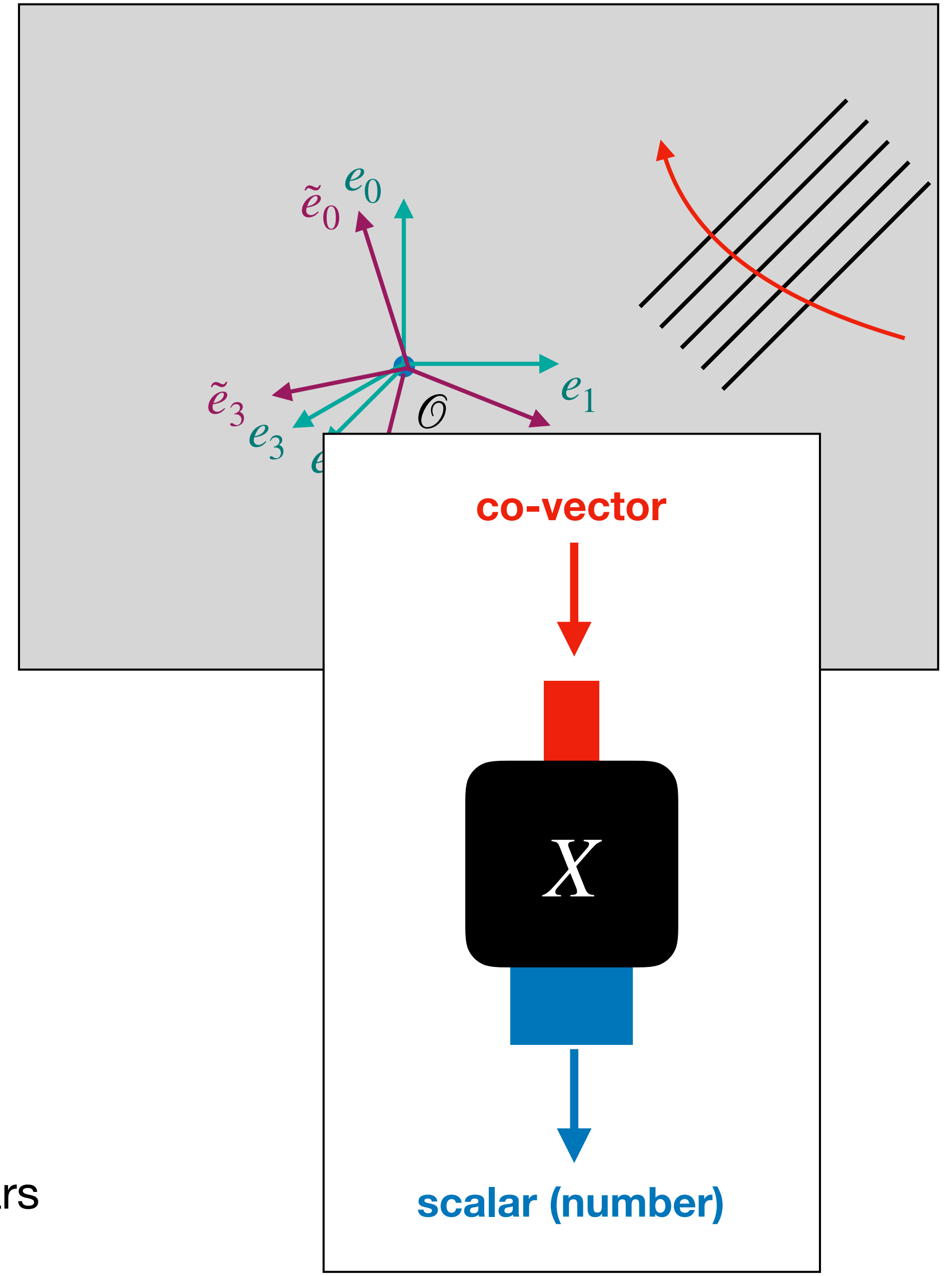


Vector - co-vector duality

$$\xi(X) = \xi_\mu \, X^\mu$$

Vector = linear function of co-vectors, yielding scalars

Tensor algebra

Tensors

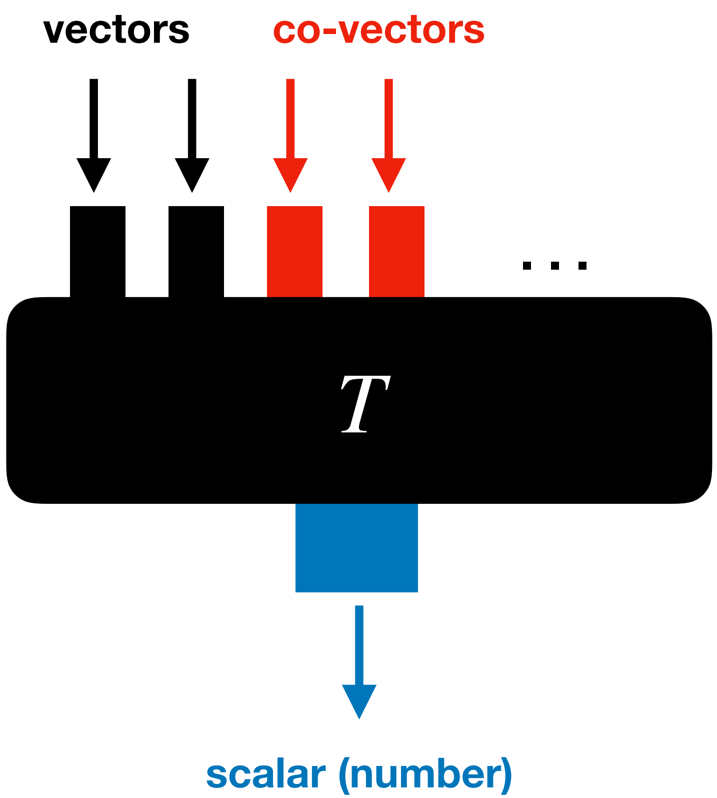
Tensor = multi-linear function of vectors and co-vectors, yielding scalars

$$T(X, Y, \xi, \mu, \ldots)$$

valence $\begin{pmatrix} k \\ l \end{pmatrix}$

co-vectors

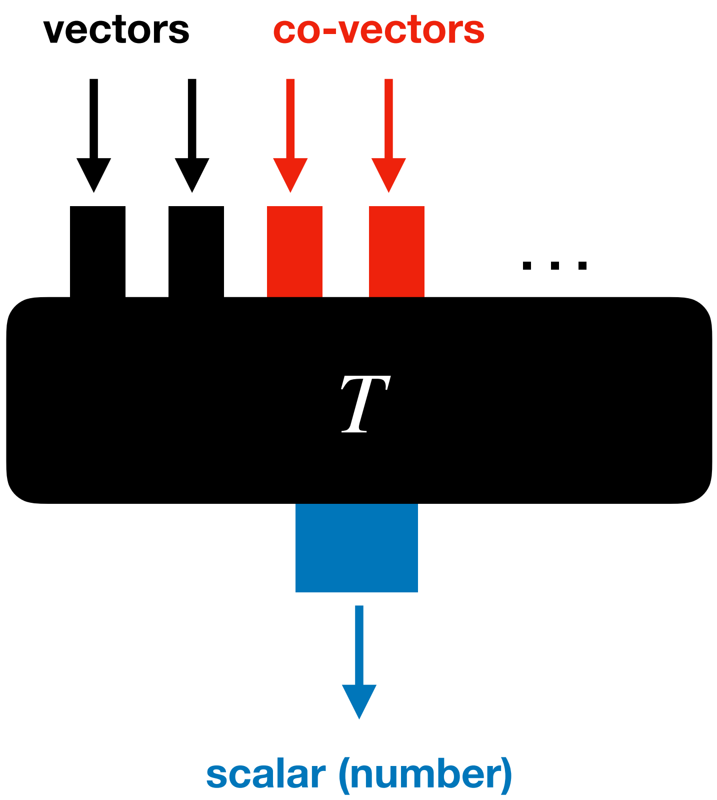
vectors



Tensor algebra

Tensors

Tensor = multi-linear function of vectors and co-vectors, yielding scalars



$$T(X, Y, \xi, \mu, \dots)$$

valence $\begin{pmatrix} k \\ l \end{pmatrix}$

co-vectors

vectors

Pick $\{e_\mu\}$ (and thus also $\{\omega^\mu\}$)

$$T(X, Y, \xi, \mu, \dots) = T_{\alpha\beta}{}^{\gamma\delta\dots} X^\alpha Y^\beta \xi_\gamma \mu_\delta \dots$$

$$T_{\alpha\beta}{}^{\gamma\delta\dots} := T(e_\alpha, e_\beta, \omega^\gamma, \omega^\delta, \dots)$$

$$\{e_\mu\} \to \{\tilde{e}_\mu\}, \quad \tilde{e}_\mu = (\Lambda^{-1})^\nu{}_\mu e_\nu$$

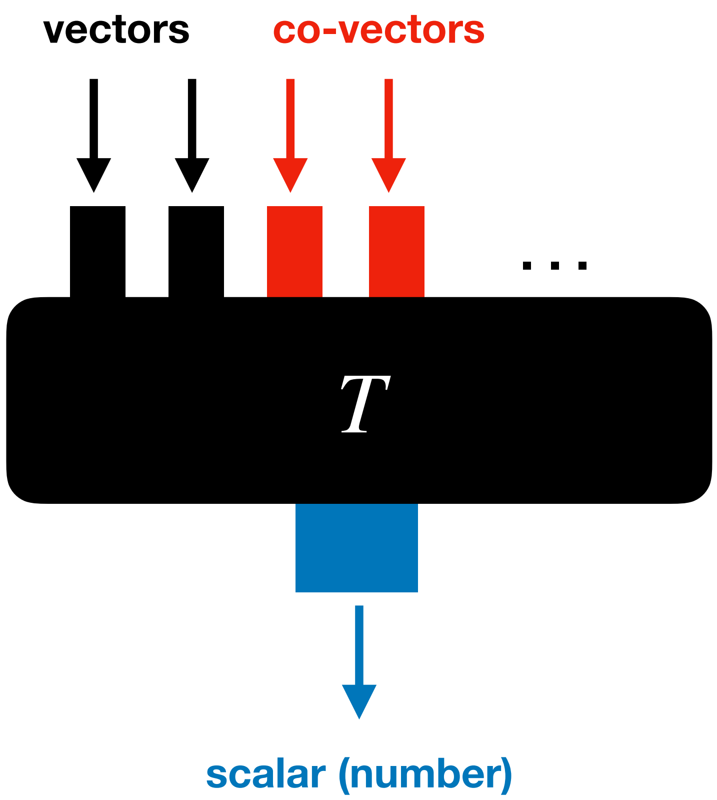
$$T_{\alpha\beta}{}^{\gamma\delta\dots} \to \tilde{T}_{\alpha\beta}{}^{\gamma\delta\dots} = T_{\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}\bar{\delta}\dots} \left(\Lambda^{-1}\right)^{\bar{\alpha}}{}_\alpha \left(\Lambda^{-1}\right)^{\bar{\beta}}{}_\beta \Lambda^\gamma{}_{\bar{\gamma}} \Lambda^\delta{}_{\bar{\delta}} \dots$$

Tensors

Tensor = multi-linear function of
vectors and co-vectors, yielding scalars

$T(X, Y, \xi, \mu, \dots)$

valence $\binom{k}{l}$ — k: # co-vectors, l: # vectors



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particular examples:

$\binom{0}{0}$ - scalar f

$\binom{1}{0}$ - vector X^μ

$\binom{0}{1}$ - co-vector ξ_μ

$$\{e_\mu\} \to \{\tilde e_\mu\}, \quad \tilde e_\mu = (\Lambda^{-1})^\nu{}_\mu e_\nu$$

$$T_{\alpha\beta}{}^{\gamma\delta\dots} \to \tilde T_{\alpha\beta}{}^{\gamma\delta\dots} = T_{\bar\alpha\bar\beta}{}^{\bar\gamma\bar\delta\dots} \left(\Lambda^{-1}\right)^{\bar\alpha}{}_\alpha \left(\Lambda^{-1}\right)^{\bar\beta}{}_\beta \Lambda^\gamma{}_{\bar\gamma} \Lambda^\delta{}_{\bar\delta} \dots$$

Tensor algebra

tensor
(abstract object)

frame / basis

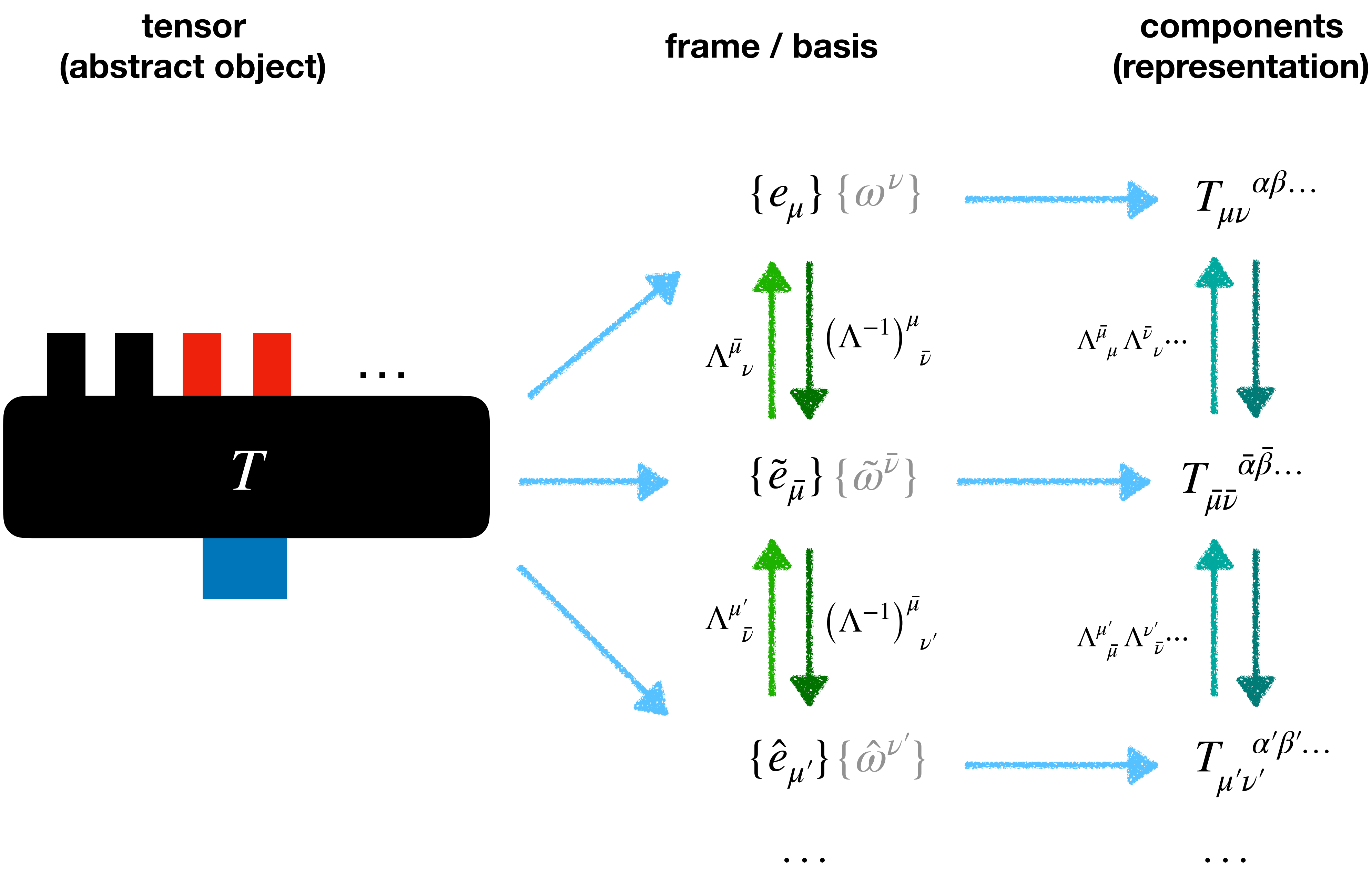
components
(representation)

$$\{e_\mu\}\ \{\omega^\nu\} \longrightarrow T_{\mu\nu}{}^{\alpha\beta\dots}$$

Tensor algebra

tensor (abstract object)	frame / basis	components (representation)
	$\{e_\mu\}\ \{\omega^\nu\}$	$T_{\mu\nu}{}^{\alpha\beta\ldots}$
	$\{\tilde{e}_{\bar{\mu}}\}\ \{\tilde{\omega}^{\bar{\nu}}\}$	$T_{\bar{\mu}\bar{\nu}}{}^{\bar{\alpha}\bar{\beta}\ldots}$
	$\{\hat{e}_{\mu'}\}\ \{\hat{\omega}^{\nu'}\}$	$T_{\mu'\nu'}{}^{\alpha'\beta'\ldots}$
	$\ldots$	$\ldots$

Tensor algebra



Tensor algebra

Making new tensors from tensors (in a frame/basis independent way)

Tensor algebra

Making new tensors from tensors (in a frame/basis independent way)

linear combinations

$$T^{\alpha}{}_{\beta\gamma\mu} = c_1 A^{\alpha}{}_{\beta\gamma\mu} + c_2 B^{\alpha}{}_{\beta\gamma\mu}$$

$$\begin{pmatrix} k \\ l \end{pmatrix}, \begin{pmatrix} k \\ l \end{pmatrix} \longrightarrow \begin{pmatrix} k \\ l \end{pmatrix}$$

Tensor algebra

Making new tensors from tensors (in a frame/basis independent way)

linear combinations

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$$\binom{k}{l}, \binom{k}{l} \longrightarrow \binom{k}{l}$$

permuting indices

$$S_{\mu}{}^{\alpha}{}_{\gamma\beta} = T^{\alpha}{}_{\beta\gamma\mu}$$

$$\binom{k}{l} \longrightarrow \binom{k}{l}$$

Tensor algebra

Making new tensors from tensors (in a frame/basis independent way)

linear combinations

$$T^{\alpha}{}_{\beta\gamma\mu} = c_1\, A^{\alpha}{}_{\beta\gamma\mu} + c_2\, B^{\alpha}{}_{\beta\gamma\mu}$$

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$$S_{\mu}{}^{\alpha}{}_{\gamma\beta} = T^{\alpha}{}_{\beta\gamma\mu}$$

$$\binom{k}{l} \longrightarrow \binom{k}{l}$$

tensor product

$$S^{\mu}{}_{\nu\alpha\beta} = T^{\mu}{}_{\nu}\, Q_{\alpha\beta}$$

$$\binom{k_1}{l_1}, \binom{k_2}{l_2} \longrightarrow \binom{k_1 + k_2}{l_1 + l_2}$$

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contraction

$$U_{\beta\mu} = T^{\alpha}{}_{\beta\alpha\mu} = T^{0}{}_{\beta 0\mu} + T^{1}{}_{\beta 1\mu} + T^{2}{}_{\beta 2\mu} + T^{3}{}_{\beta 3\mu}$$

$$\binom{k}{l} \longrightarrow \binom{k-1}{l-1}$$

Tensor algebra

Index permutation symmetries

Tensor algebra

Index permutation symmetries

Tensor symmetric in 2 indices

$$T_{\alpha\beta} = T_{\beta\alpha}$$

$$T(X, Y) = T(Y, X)$$

Tensor algebra

Index permutation symmetries

Tensor symmetric in 2 indices

$$T_{\alpha\beta} = T_{\beta\alpha}$$

$$T(X, Y) = T(Y, X)$$

Tensor anti-symmetric (skew-symmetric) in 2 indices

$$T_{\alpha\beta} = -T_{\beta\alpha}$$

$$T(X, Y) = -T(Y, X)$$

More complicated multi-index symmetries also possible

Tensor algebra

Special tensor:

unit tensor $\mathbf{1}$ $\quad$ $\delta^{\mu}{}_{\nu} = \begin{cases} 1 & \text{if} \quad \alpha = \beta \\ 0 & \text{if} \quad \alpha \neq \beta \end{cases}$ $\quad$ has the same components in all frames

$\delta^{\mu}{}_{\nu} X^{\nu} = X^{\mu}$ $\qquad$ $\delta^{\mu}{}_{\nu} \xi_{\mu} = \xi_{\nu}$

Tensor algebra

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Metric tensor:

scalar product $X \cdot Y$ is bi-linear

Tensor algebra

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$$\delta^{\mu}{}_{\nu} X^{\nu} = X^{\mu} \qquad \delta^{\mu}{}_{\nu} \xi_{\mu} = \xi_{\nu}$$

Metric tensor:

scalar product $X \cdot Y$ is bi-linear $\Longrightarrow$ it is a $\binom{0}{2}$-tensor called the ***metric tensor*** g

$$X \cdot Y = g(X, Y) = g_{\mu\nu} X^{\mu} Y^{\nu}$$

Symmetric tensor $\qquad g_{\mu\nu} = g_{\nu\mu}$

In an inertial frame $\qquad g_{\mu\nu} = \eta_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Index raising and lowering

Define the inverse metric tensor

$$g^{\mu\nu}\, g_{\nu\alpha} = \delta^{\mu}{}_{\alpha}$$

inverse matrix to $g_{\nu\alpha}$

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X^{μ} - vector $\qquad X^{\mu}\, g_{\mu\nu}$

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Duality:

X^{μ} - vector $\quad X^{\mu}\, g_{\mu\nu} =: X_{\nu}$

$$X_{\mu}\, g^{\mu\nu} = X^{\alpha}\, g_{\alpha\mu}\, g^{\mu\nu} = X^{\alpha}\, \delta^{\nu}{}_{\alpha} = X^{\nu}$$

ξ_{μ} - co-vector $\quad \xi_{\mu}\, g^{\mu\nu} =: \xi^{\nu}$

$$\xi^{\mu}\, g_{\mu\nu} = \xi_{\alpha}\, g^{\alpha\mu}\, g_{\mu\nu} = \xi_{\alpha}\, \delta^{\alpha}{}_{\nu} = \xi_{\nu}$$

Index raising and lowering

Remarks

We can use any of metric's indices

$$X_\mu = X^\alpha g_{\mu\alpha} = X^\alpha g_{\alpha\mu} \qquad \xi^\mu = \xi_\beta g^{\mu\beta} = \xi_\beta g^{\beta\mu}$$

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Co-vectors have their product too!

$$\xi \cdot \zeta = \xi_\mu \zeta_\nu g^{\mu\nu} = \xi^\mu \zeta^\nu g_{\mu\nu}$$

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This works for tensors too:

$$T_{\mu\sigma\alpha\ldots} = g_{\nu\sigma} T_\mu{}^\nu{}_{\alpha\ldots} \qquad \begin{pmatrix} k \\ l \end{pmatrix} \longrightarrow \begin{pmatrix} k-1 \\ l+1 \end{pmatrix}$$

$$T^{\rho\nu}{}_{\alpha\ldots} = g^{\rho\mu} T_\mu{}^\nu{}_{\alpha\ldots} \qquad \begin{pmatrix} k \\ l \end{pmatrix} \longrightarrow \begin{pmatrix} k+1 \\ l-1 \end{pmatrix}$$

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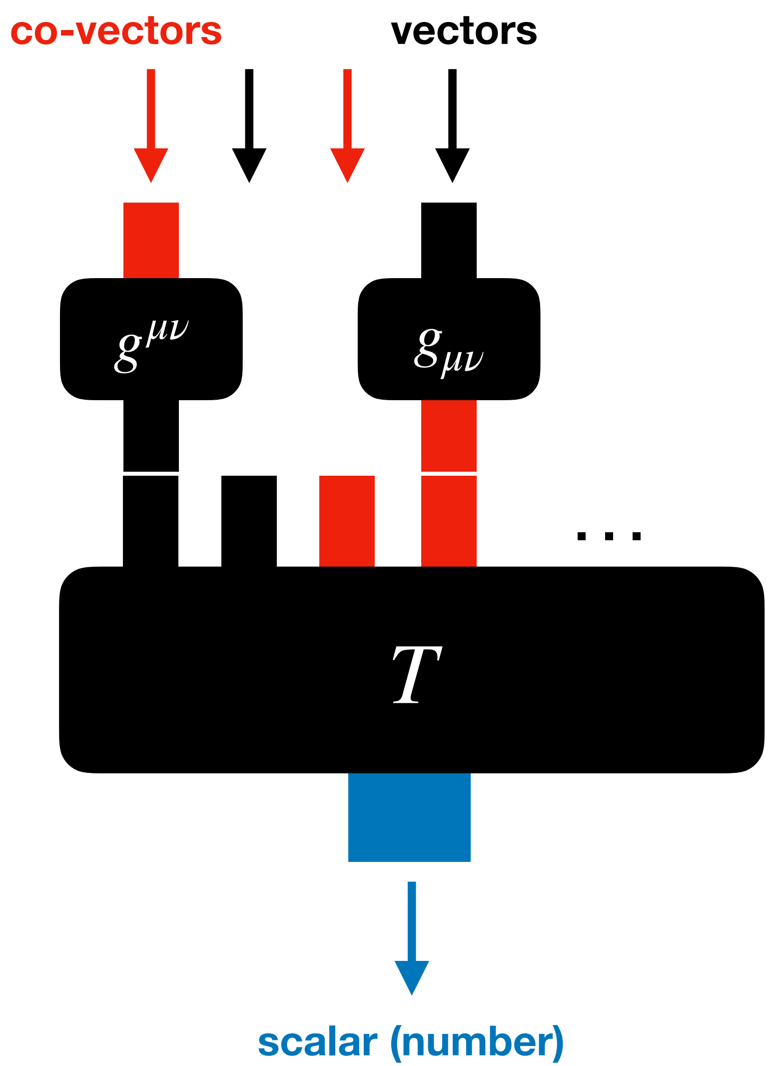

Index raising and lowering

Remarks

We can now trace over any pair of indices

$$T_{\alpha\beta} \qquad T^{\beta}{}_{\beta} = T_{\alpha\beta}\, g^{\alpha\beta} \qquad \begin{pmatrix} k \\ l \end{pmatrix} \longrightarrow \begin{pmatrix} k \\ l-2 \end{pmatrix}$$

$$S^{\mu}{}_{\nu}{}^{\alpha\beta} \qquad S^{\mu}{}_{\nu}{}^{\alpha}{}_{\mu} = S^{\mu}{}_{\nu}{}^{\alpha\beta}\, g_{\mu\beta} \qquad \begin{pmatrix} k \\ l \end{pmatrix} \longrightarrow \begin{pmatrix} k-2 \\ l \end{pmatrix}$$

Index raising and lowering

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We can now trace over any pair of indices

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$$S^{\mu}{}_{\nu}{}^{\alpha\beta} \qquad S^{\mu}{}_{\nu}{}^{\alpha}{}_{\mu} = S^{\mu}{}_{\nu}{}^{\alpha\beta}\, g_{\mu\beta} \qquad \binom{k}{l} \longrightarrow \binom{k-2}{l}$$

End of lecture 2