Schwarzschild solution

Geodesics

$$g = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

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$$\frac{d^2t}{d\sigma^2} + \frac{2GM}{r(r-2GM)} \frac{dr}{d\sigma} \frac{dt}{d\sigma} = 0$$

$$\frac{d^2r}{d\sigma^2} + \frac{GM(r-2GM)}{r^3} \left(\frac{dt}{d\sigma}\right)^2 - \frac{GM}{r(r-2GM)} \left(\frac{dr}{d\sigma}\right)^2 - (r-2GM)\left(\left(\frac{d\theta}{d\sigma}\right)^2 + \sin^2\theta \left(\frac{d\phi}{d\sigma}\right)^2\right) = 0$$

$$\frac{d^2\theta}{d\sigma^2} + \frac{2}{r} \frac{d\theta}{d\sigma} \frac{dr}{d\sigma} - \sin\theta\, \cos\theta \left(\frac{d\phi}{d\sigma}\right)^2 = 0$$

$$\frac{d^2\phi}{d\sigma^2} + \frac{2}{r} \frac{d\phi}{d\sigma} \frac{dr}{d\sigma} + 2\frac{\cos\theta}{\sin\theta} \frac{d\theta}{d\sigma} \frac{d\phi}{d\sigma} = 0$$

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$$\theta(\sigma_0) = \frac{\pi}{2}, \qquad \left.\frac{d\theta}{d\sigma}\right|_{\sigma_0} = 0$$

$$\Longrightarrow \theta = \frac{\pi}{2} = \text{const}$$

motion is always planar

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typical for motion in spherically symmetric situations

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we can always rotate the coordinate system to make the motion planar in $\theta = \frac{\pi}{2}$ (or, equivalently, $z = 0$)

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$$L_x = p^\mu \Phi^x_\mu$$

$$L_y = p^\mu \Phi^y_\mu$$

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Idea: reduce to the problem of 1D motion in radial direction with the help of conserved quantities