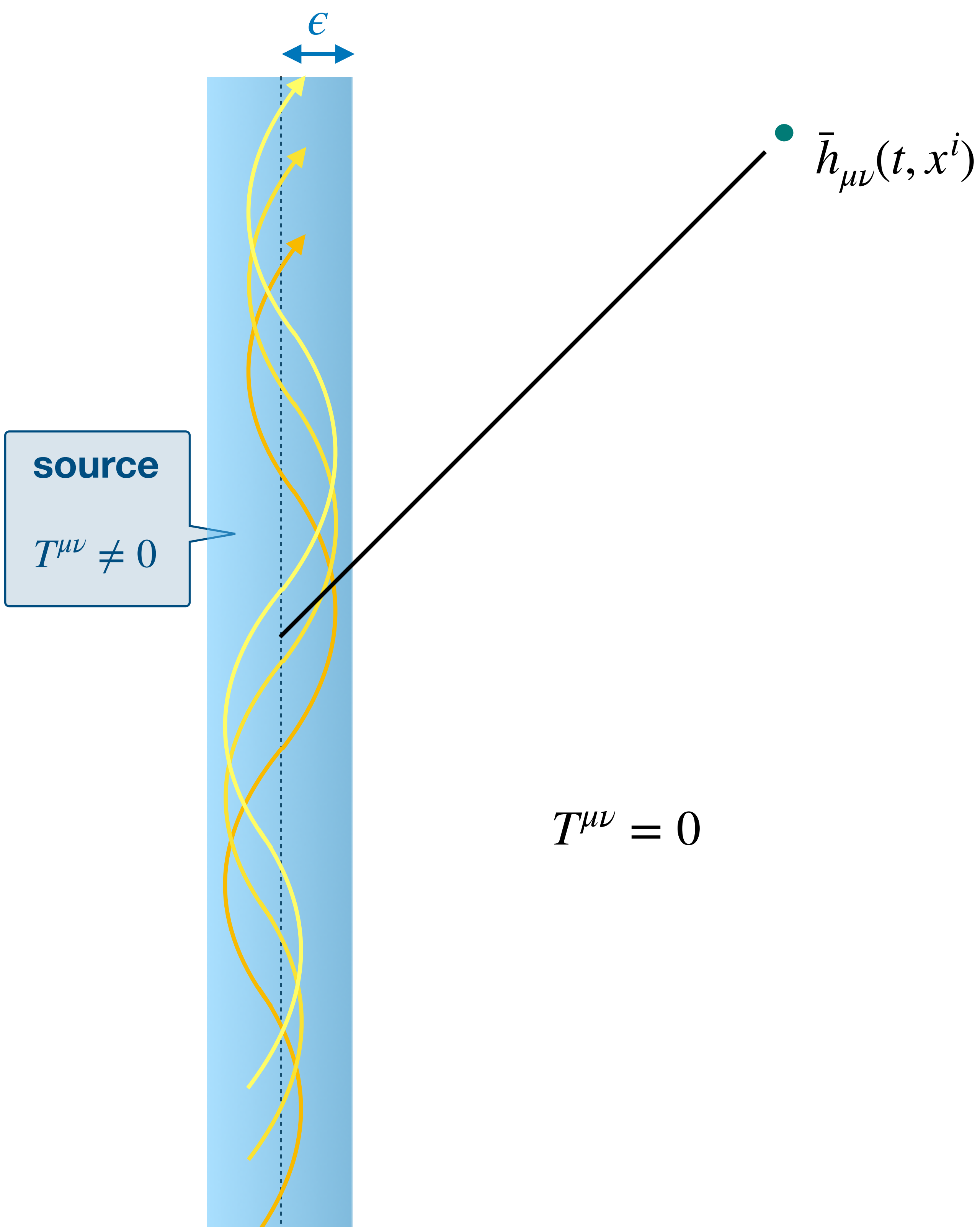
Gravitational waves

Generating gravitational waves



Assumptions

Gravitational waves

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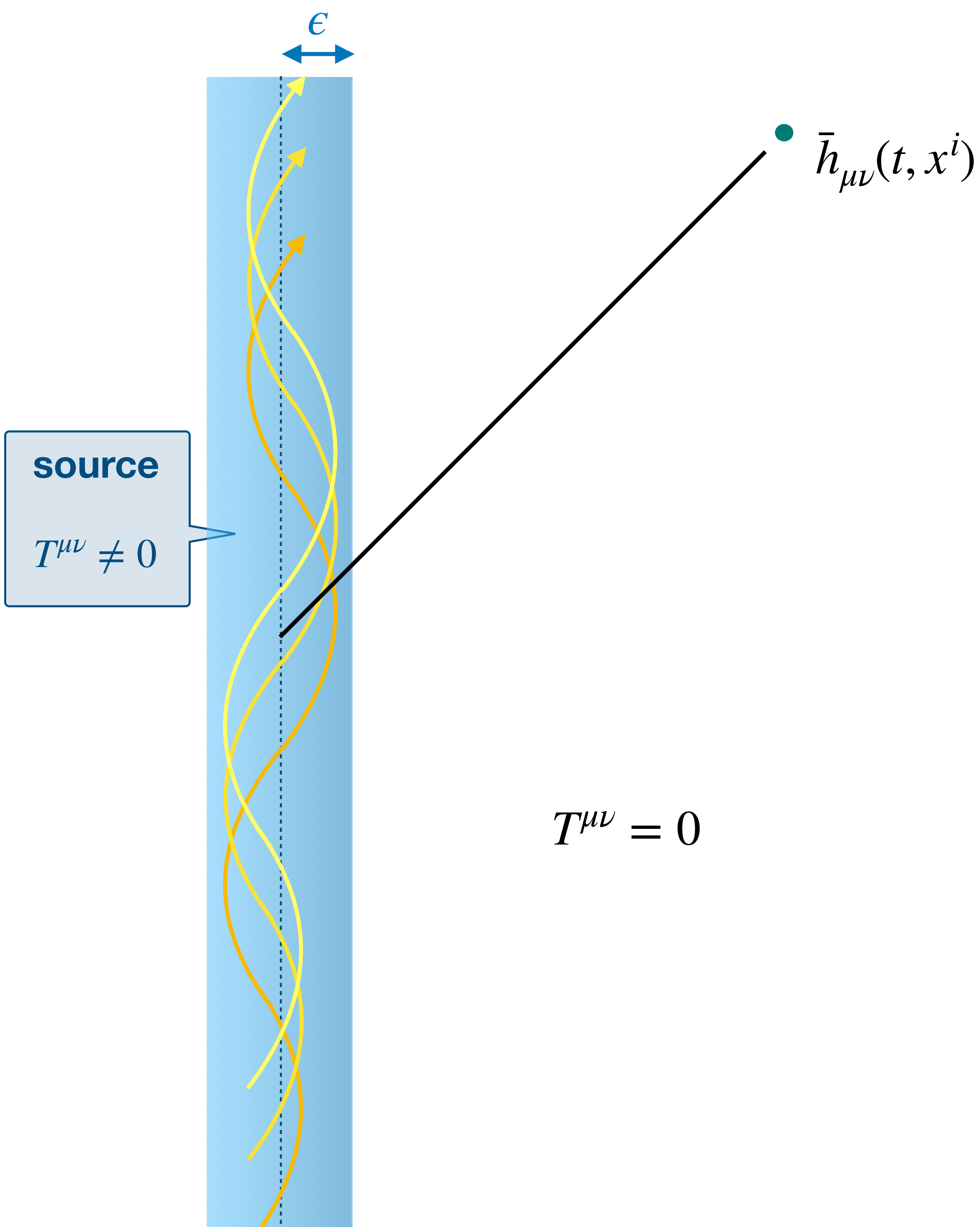

Assumptions

- vacuum outside the source's world-tube

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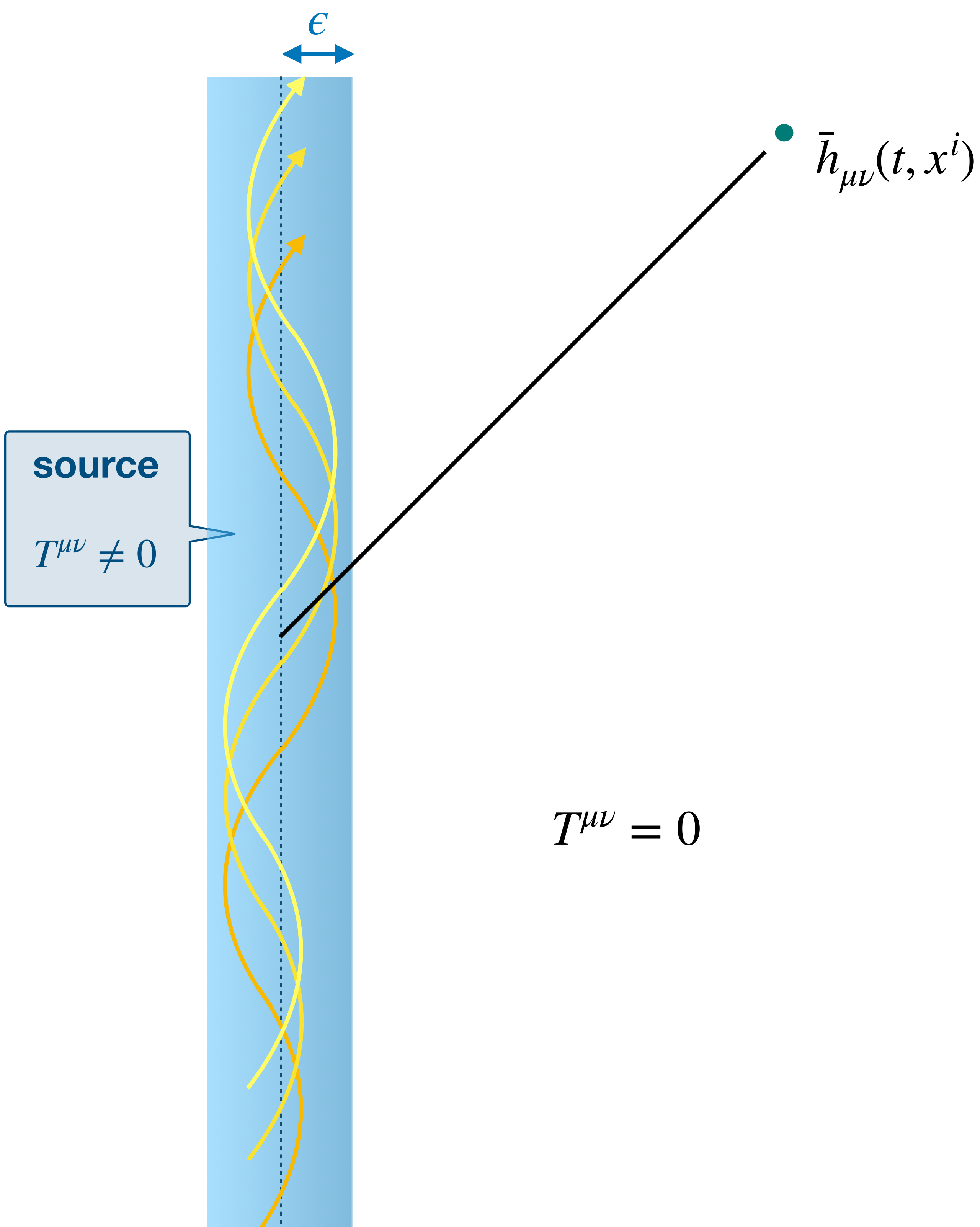
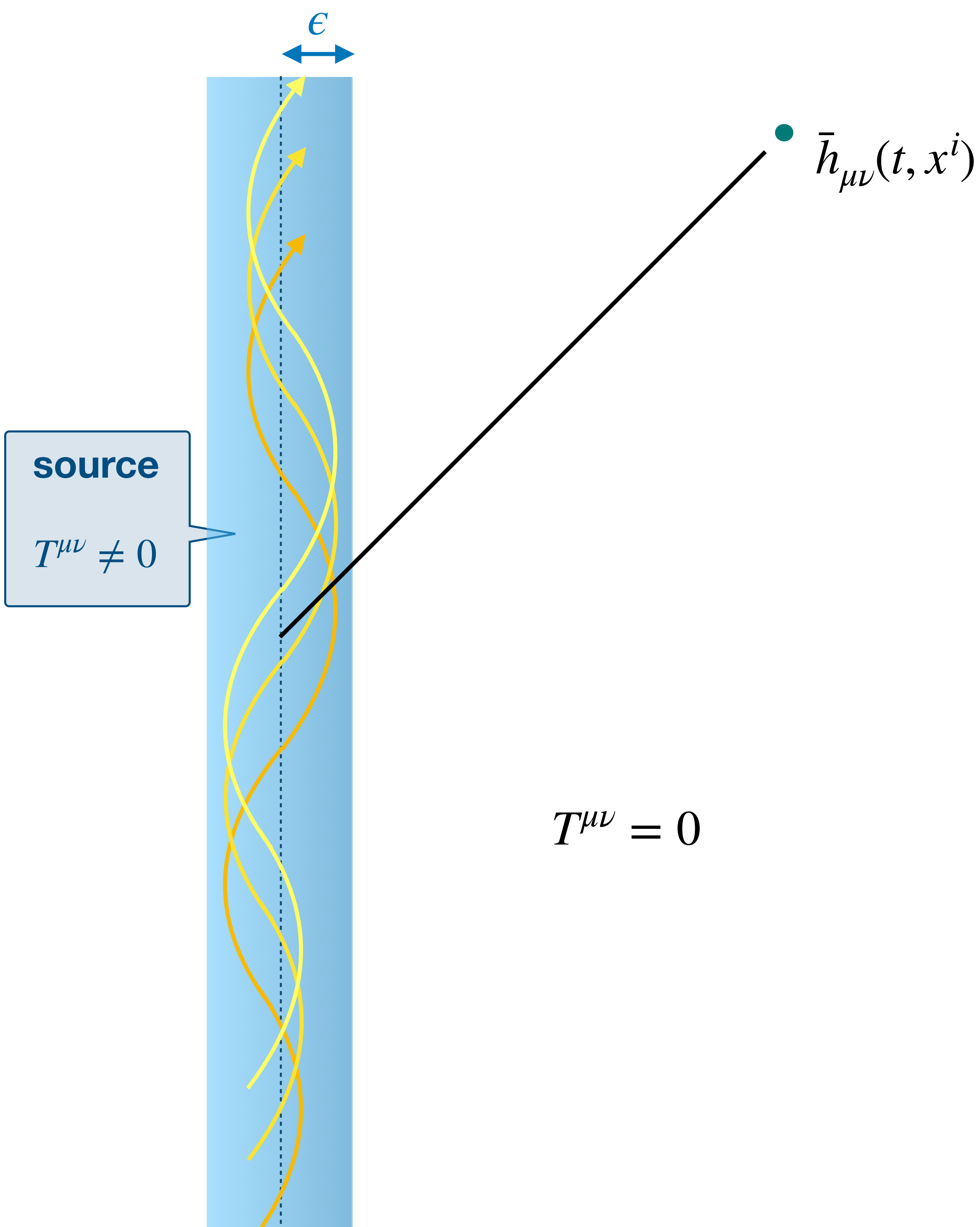


Assumptions

- vacuum outside the source's world-tube
- size of the source much smaller than wavelength $\epsilon \ll \dfrac{2\pi}{\omega}$

Gravitational waves

Generating gravitational waves



Assumptions

- vacuum outside the source's world-tube
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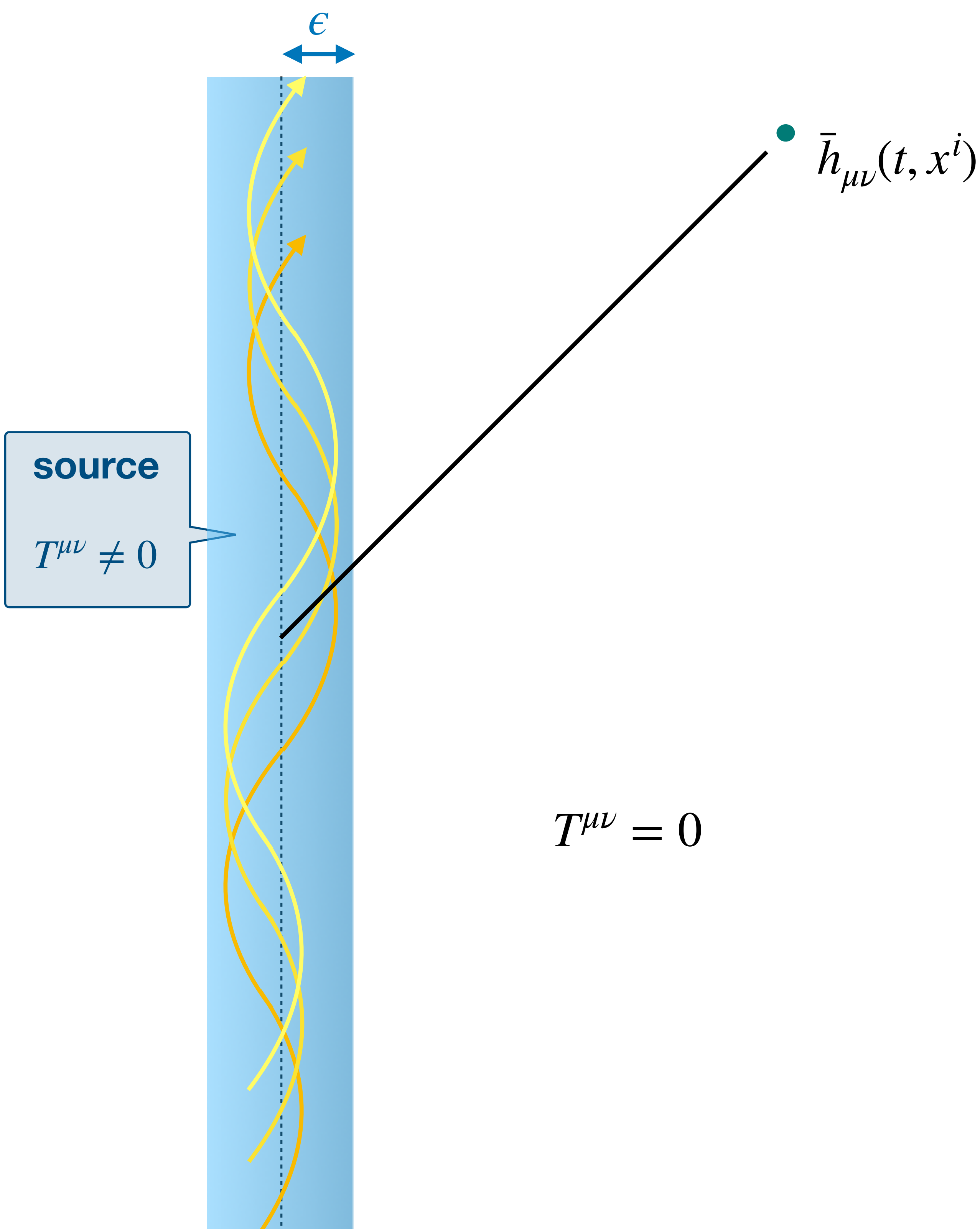
  $$\epsilon \ll \frac{2\pi}{\omega}$$
- slow motions within the source
  $$T_{\mu\nu}(t + \Delta t, \vec{x}) \approx T_{\mu\nu}(t, \vec{x})$$

for $\Delta t \approx \epsilon$

Gravitational waves

Generating gravitational waves



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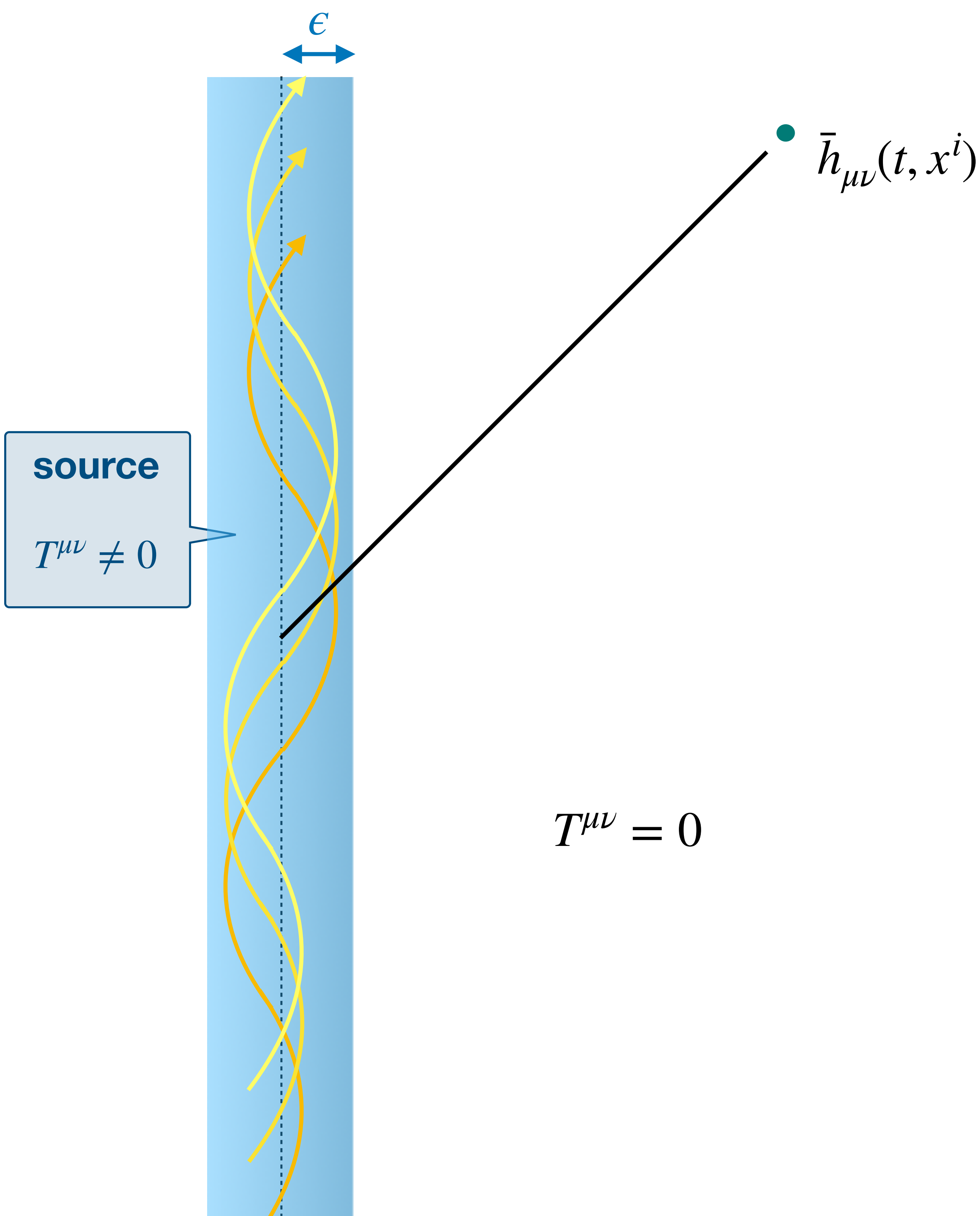

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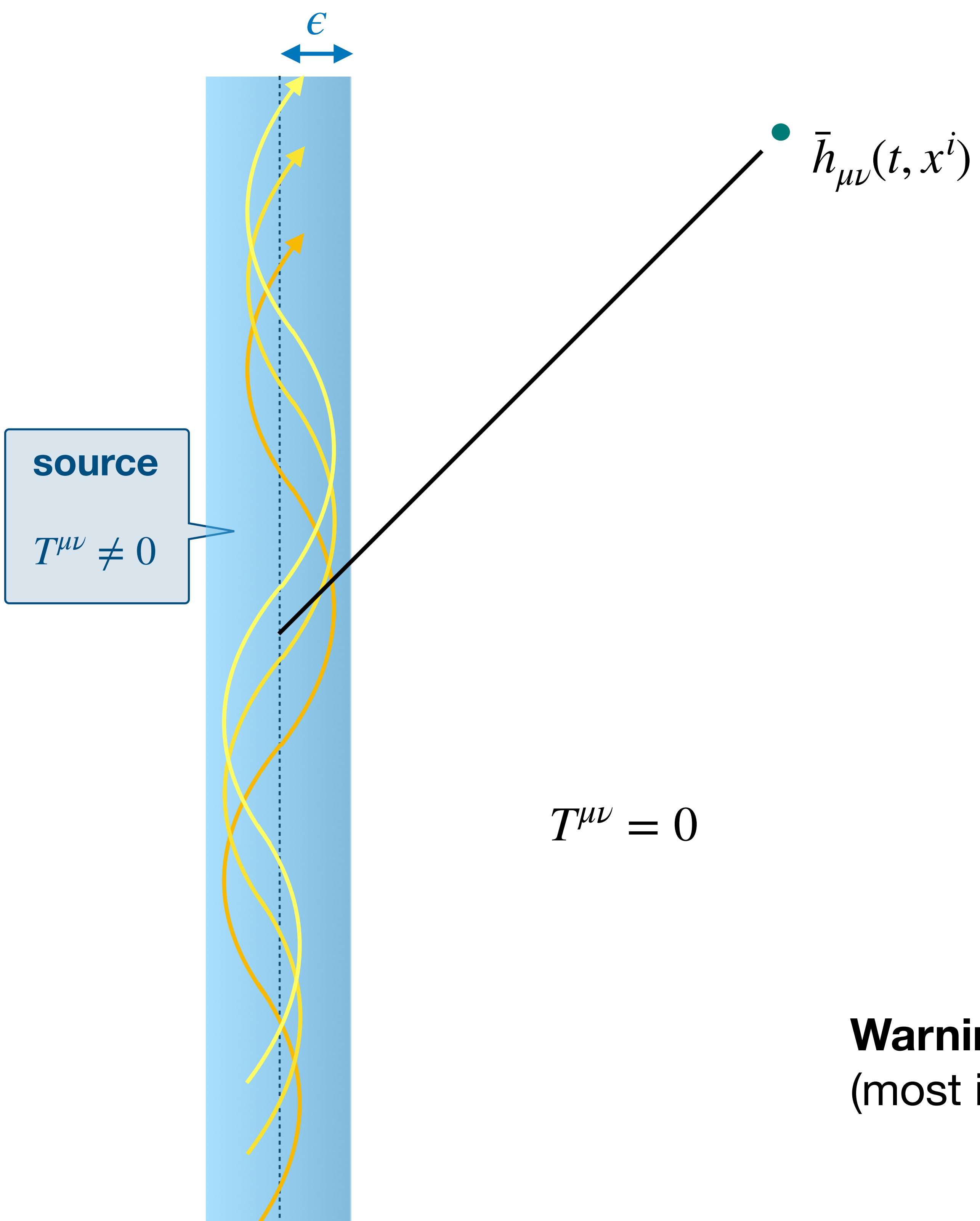
Gravitational waves

Generating gravitational waves



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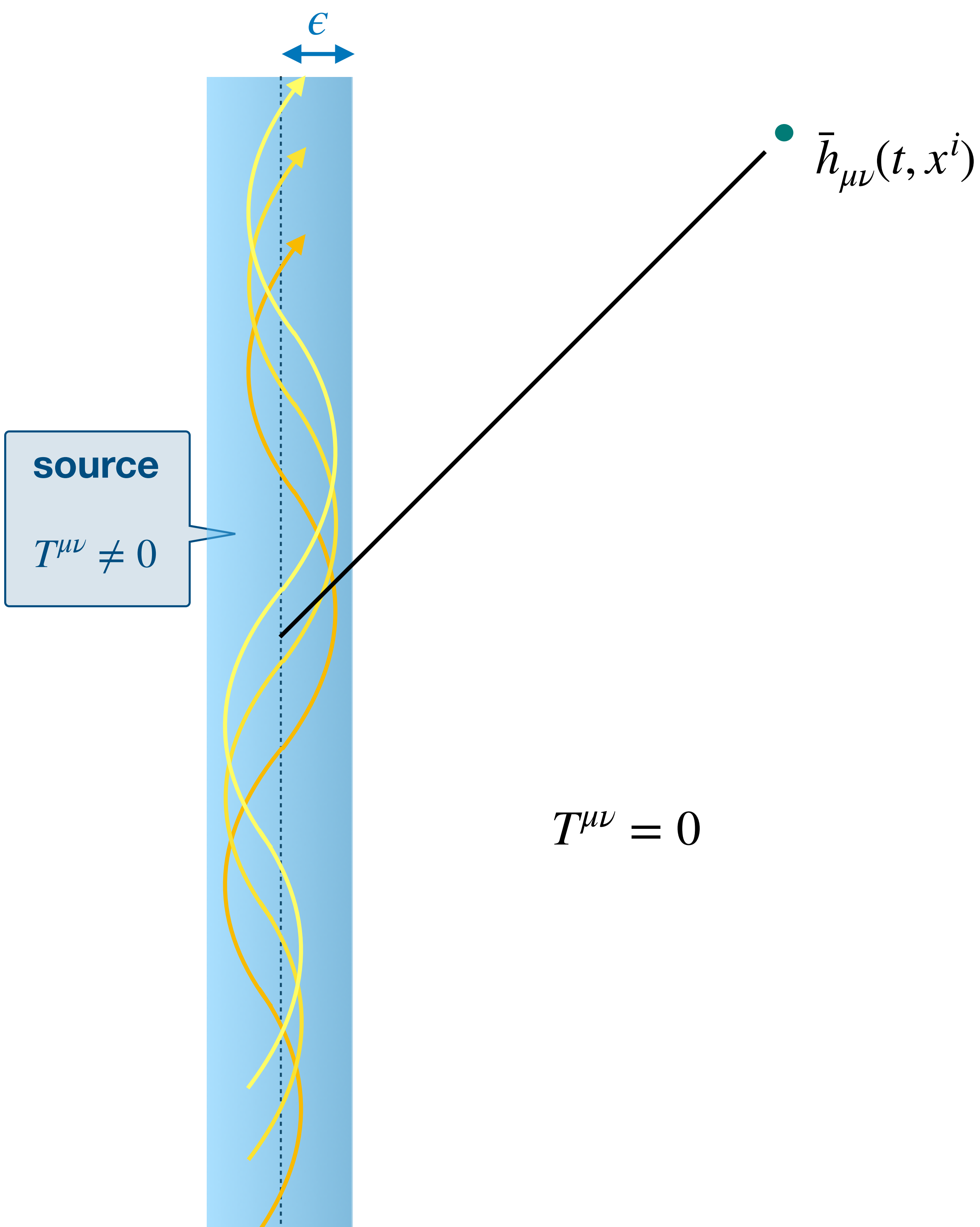
Warning: not a good approximation for coalescing binaries (most interesting astrophysical sources nowadays!)

Gravitational waves

Generating gravitational waves



Quadrupole formula

$$h_{ij}^{TT}(t, \vec{x}) = \frac{2G}{|\vec{x}|} \left(\frac{d^2}{dt^2} \mathcal{I}_{kl}(t - |\vec{x}|) \right) \cdot \cdot \left(P^k{}_i P^l{}_j - \frac{1}{2} P^{kl} P_{ij} \right)$$

where

- reduced quadrupole moment

$$\mathcal{I}_{kl} = \int d^3x \left(x^k x^l - \frac{1}{3} x^i x_i \delta^{kl} \right) T^{00}(t, x^i)$$

- transverse projection operator

$$P^i{}_j(\vec{x}) = \delta^i{}_j - |\vec{x}|^{-2} (x^i x_j)$$

- projection to the TT tensor

$$P^k{}_i P^l{}_j - \frac{1}{2} P^{kl} P_{ij}$$

Gravitational waves

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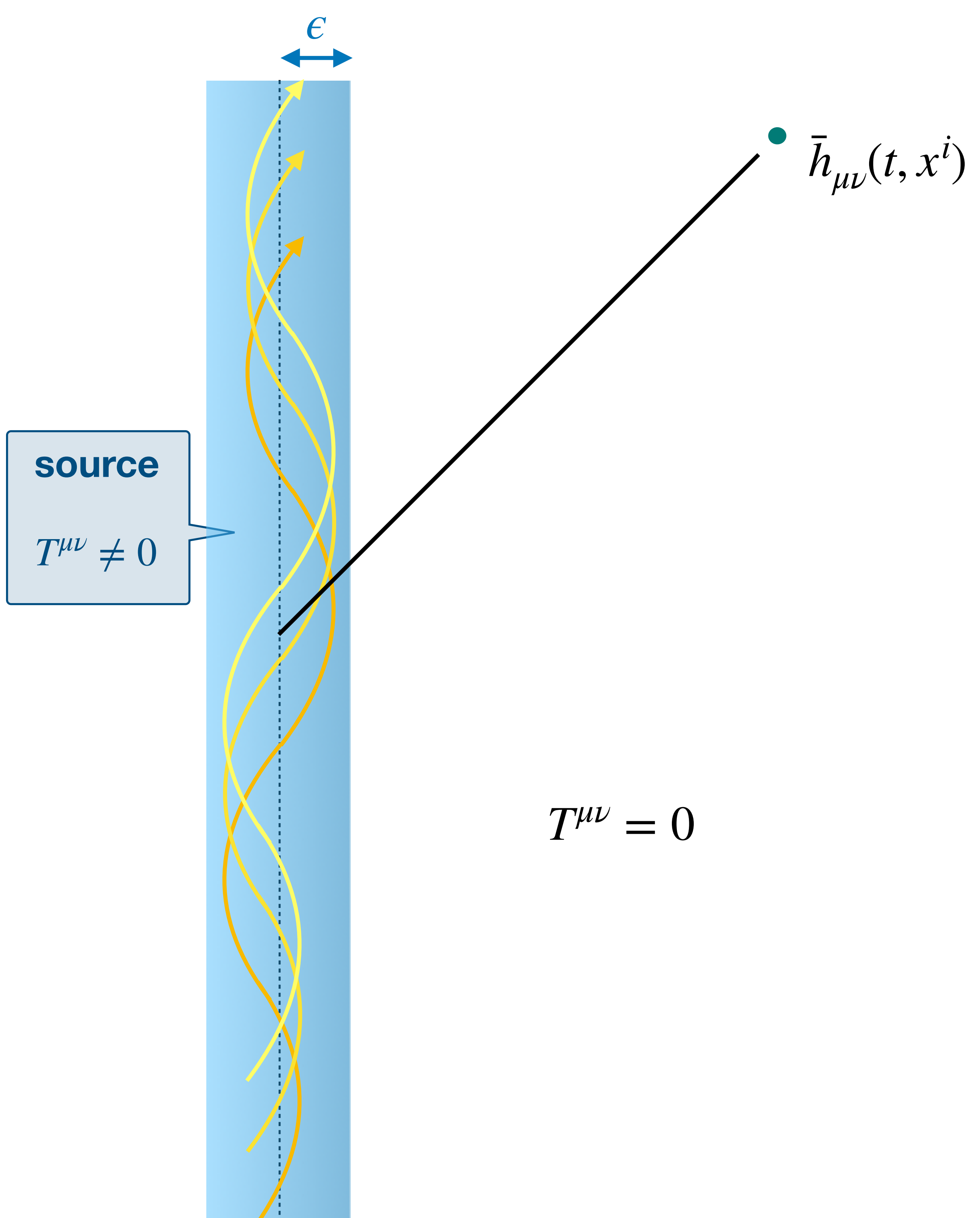


Emitted energy

$$h_{ij}^{TT}(t, \vec{x}) = \frac{2G}{|\vec{x}|} \left(\frac{d^2}{dt^2} \mathcal{I}_{kl}(t - |\vec{x}|) \right) \cdot \left(P^k{}_i P^l{}_j - \frac{1}{2} P^{kl} P_{ij} \right)$$

Combining the quadrupole formula with the energy flux integrated over a large sphere yields the total power

$$P = \int_{r=R} \sin\theta \, d\theta \, d\phi \, \underbrace{(F_+ + F_\times)}_{F}$$

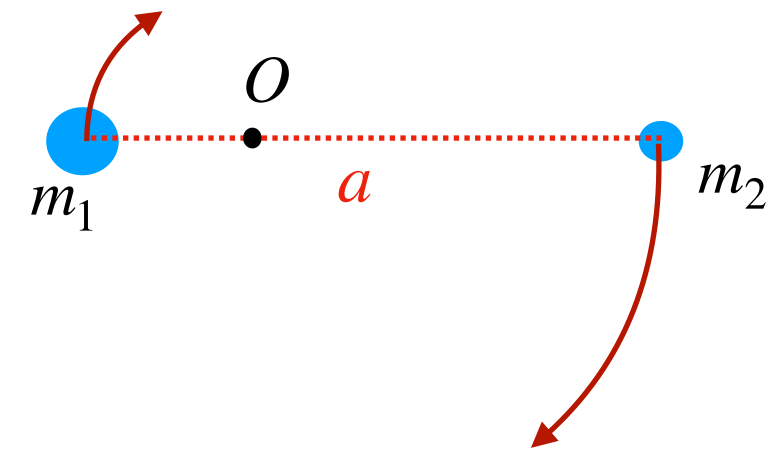
r^2

$$P = -\frac{G}{5} \left\langle \frac{d^3 \mathcal{I}_{ij}}{dt^3} \frac{d^3 \mathcal{I}^{ij}}{dt^3} \right\rangle$$

Radiating systems lose energy!

Gravitational waves

Sources of gravitational waves



Simplest possible source: 2 heavy point masses revolving around the barycenter

$$m_1 \, r_1 = m_2 \, r_2$$

$$a = r_1 + r_2$$

Gravitational waves

Sources of gravitational waves

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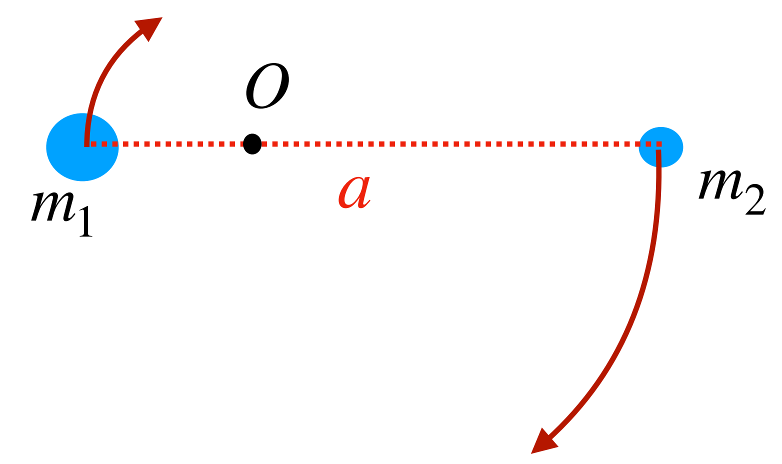

$$m_1 r_1 = m_2 r_2$$

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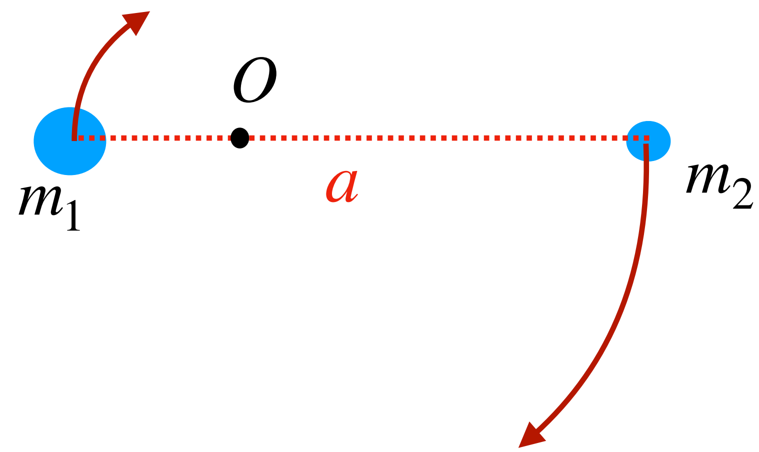
$$r_1 = \frac{m_2 a}{m_1 + m_2} \qquad r_2 = \frac{m_1 a}{m_1 + m_2}$$

Gravitational waves

Sources of gravitational waves



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$$\vec{x}_1(t) = r_1 \begin{pmatrix} \cos \Omega t \\ \sin \Omega t \\ 0 \end{pmatrix} \qquad \vec{x}_1(t) = -\, r_2 \begin{pmatrix} \cos \Omega t \\ \sin \Omega t \\ 0 \end{pmatrix}$$

Gravitational waves

Sources of gravitational waves

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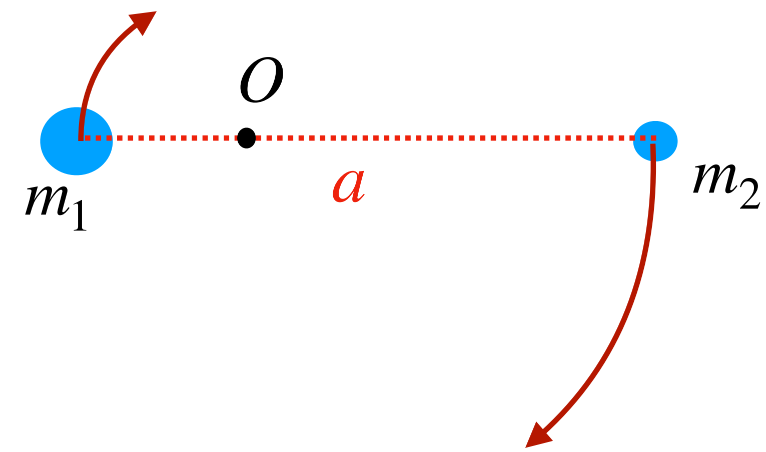

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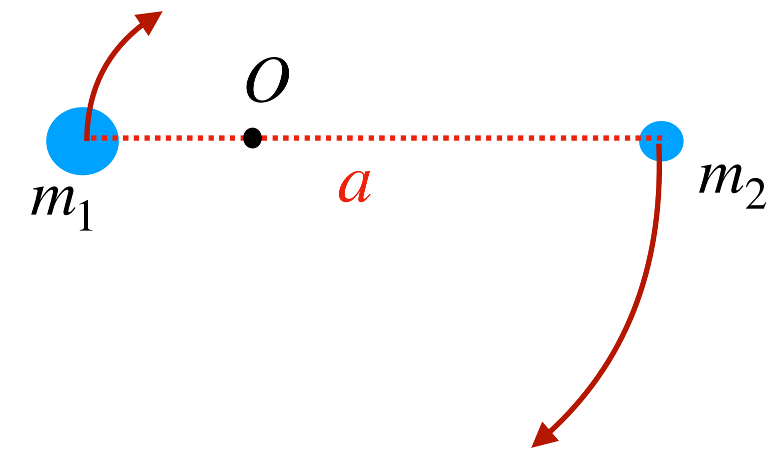
$$I_{ij} = \mu a^2 \begin{pmatrix} \cos^2 \Omega t & \cos \Omega t \sin \Omega t & 0 \\ \cos \Omega t \sin \Omega t & \sin^2 \Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathcal{I}_{ij} = \mu a^2 \begin{pmatrix} \cos^2 \Omega t - \frac{1}{3} & \cos \Omega t \sin \Omega t & 0 \\ \cos \Omega t \sin \Omega t & \sin^2 \Omega t - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Gravitational waves

Sources of gravitational waves



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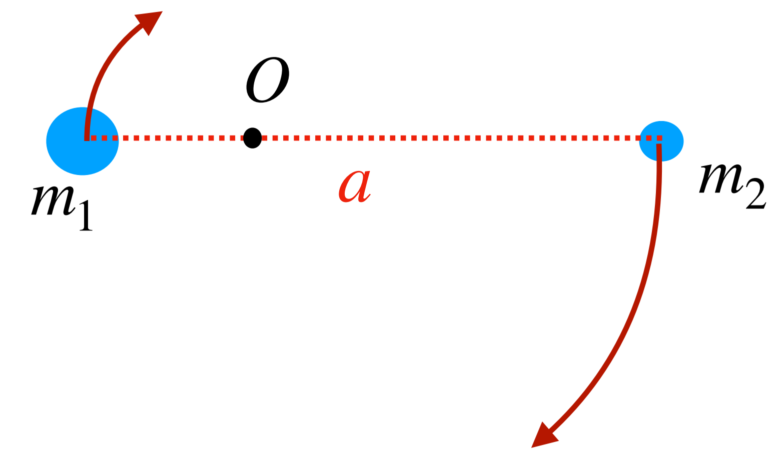
Applying trigonometric identities…

Gravitational waves

Sources of gravitational waves



$$\mathcal{I}_{ij} = \frac{\mu a^2}{2} \begin{pmatrix} \cos 2\Omega t + \frac{1}{3} & \sin 2\Omega t & 0 \\ \sin 2\Omega t & -\cos 2\Omega t + \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}$$

Applying the quadrupole formula

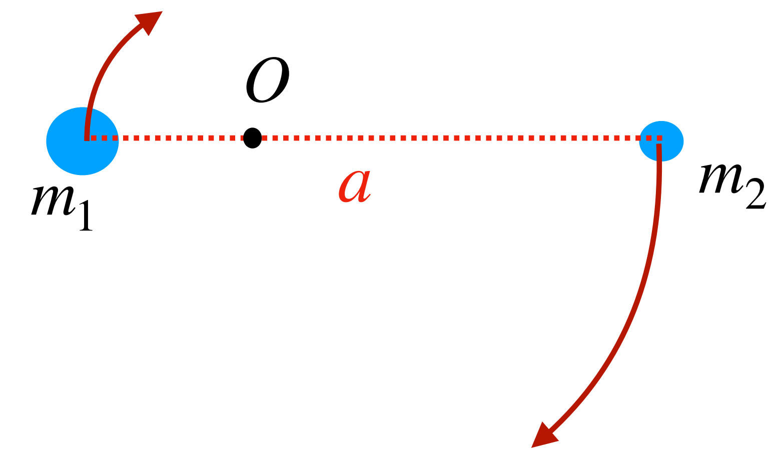
$$\bar{h}^{TT}_{ij} = -\frac{4G\mu}{r} a^2 \Omega^2 \begin{pmatrix} \cos 2\Omega t + \cancel{\frac{1}{3}} & \sin 2\Omega t & 0 \\ \sin 2\Omega t & -\cos 2\Omega t + \cancel{\frac{1}{3}} & 0 \\ 0 & 0 & 0 \, \cancel{-\frac{2}{3}} \end{pmatrix}^{TT}$$

re-introducing the speed of light…

$$h_{ij} \approx \frac{4G\mu \, a^2 \Omega^2}{r \, c^4}$$

Gravitational waves

Sources of gravitational waves



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re-introducing the speed of light…

$$h_{ij} \approx \frac{4G\mu\, a^2 \Omega^2}{r\, c^4}$$

Lab sources

$\mu = 10^3\,\mathrm{kg}$ $\qquad$ $a = 1\,\mathrm{m}$

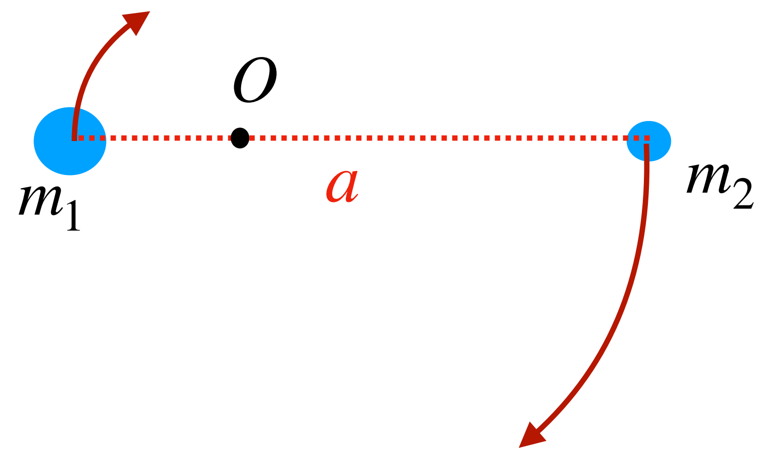
$\Omega = 1\,\mathrm{rad\,s^{-1}}$

$$h_{ij} \approx 10^{-41} \left(\frac{1\mathrm{m}}{r} \right)$$

Gravitational waves

Astrophysical sources

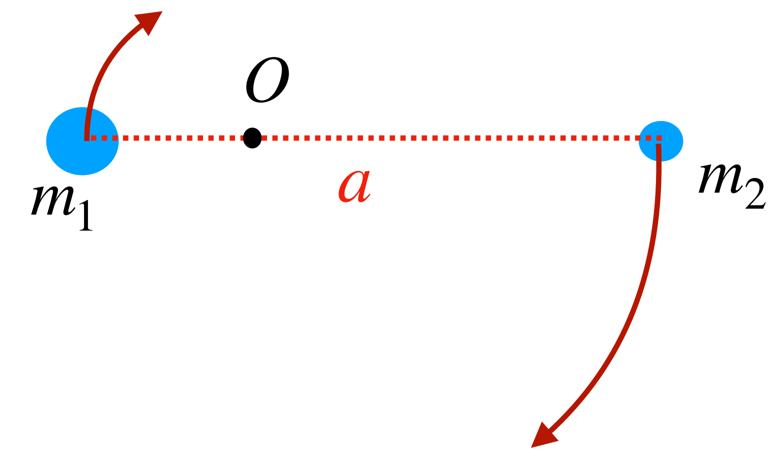
$$h_{ij} \approx \frac{4G\mu\, a^2\Omega^2}{r\, c^4}$$



for a circular Keplerian orbit $\Omega = (G\,(m_1 + m_2))^{1/2}\, a^{-3/2}$

Gravitational waves

Astrophysical sources



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$$h_{ij} \approx \frac{4G^2 m_1 m_2}{a\, r\, c^4}$$

$$h_{ij} \approx \frac{4G\mu}{r\, c^4}\, \Omega^{2/3}\, \left(G(m_1 + m_2)\right)^{2/3}$$

$$m_1 \approx m_2 \approx \mu \approx M$$

$$h_{ij} \approx \frac{(GM)^{5/3}\, \Omega^{2/3}}{c^4\, r}$$

Gravitational waves

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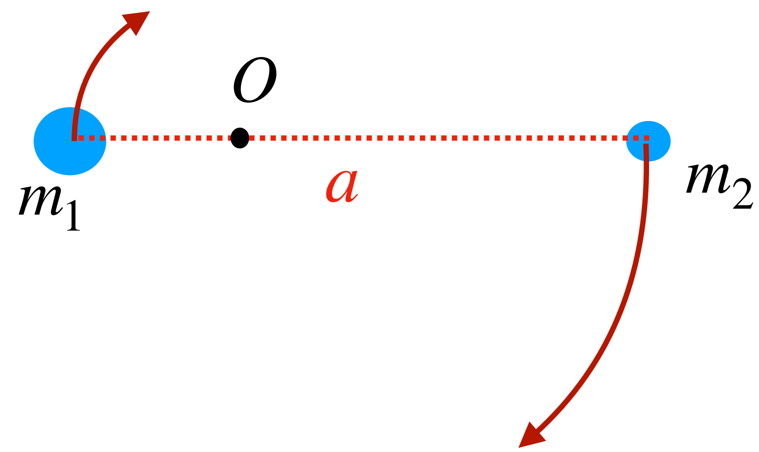

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Binary white dwarfs

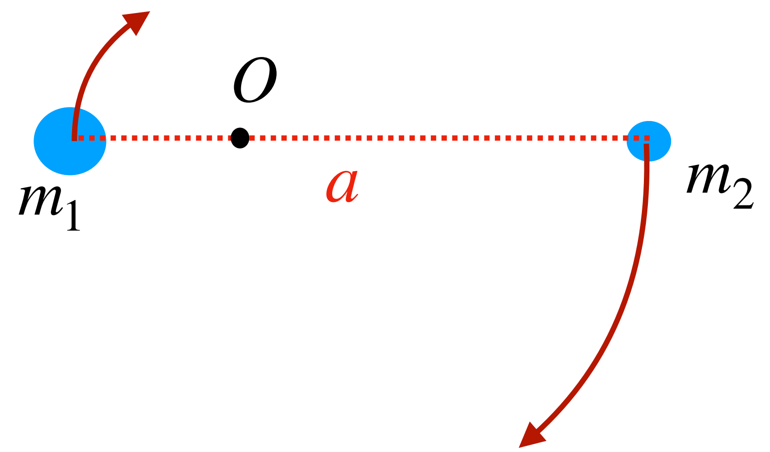
$$h_{ij} \approx 10^{-21} \left(\frac{M}{2M_\odot}\right)^{5/3} \left(\frac{1\mathrm{h}}{P}\right)^{2/3} \left(\frac{1\mathrm{kpc}}{r}\right)$$

Binary neutron stars

$$h_{ij} \approx 10^{-22} \left(\frac{M}{2.8M_\odot}\right)^{5/3} \left(\frac{0.01\mathrm{s}}{P}\right)^{2/3} \left(\frac{100\mathrm{Mpc}}{r}\right)$$

Gravitational waves

Total power of the source (luminosity)



$$L = -P = \frac{G}{5c^5}\left\langle \dddot{\mathcal{I}}_{ij}\dddot{\mathcal{I}}^{ij}\right\rangle$$

$$L = \frac{32G\,\mu^2\,a^4\,\Omega^6}{5c^5}$$

Gravitational waves

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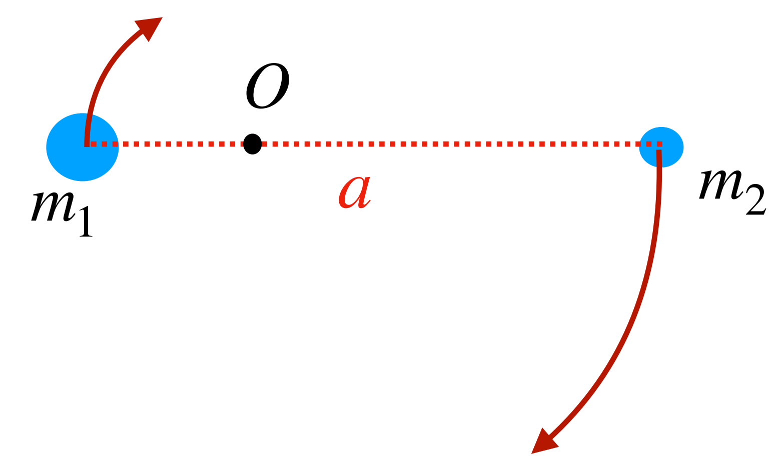

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for a circular Keplerian orbit

$$L = \frac{32}{5}\mu c^2\,\Omega\left(\frac{G\mu^{3/7}M_{\mathrm{tot}}^{4/7}\Omega}{c^3}\right)^{7/3}$$

Gravitational waves

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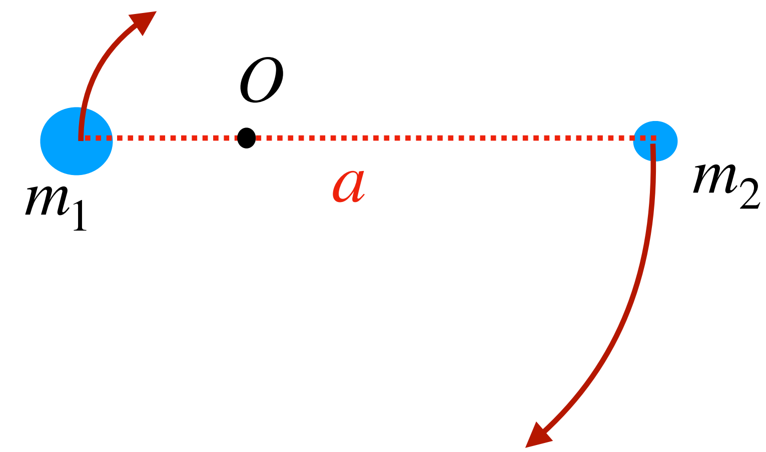

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$$L \approx Mc^2\Omega\left(\frac{GM\Omega}{c^3}\right)^{7/3} = \frac{c^5}{G}\left(\frac{GM\Omega}{c^3}\right)^{10/3}$$

„natural" unit of luminosity in GR $3.63 \cdot 10^{52}$ W

Gravitational waves

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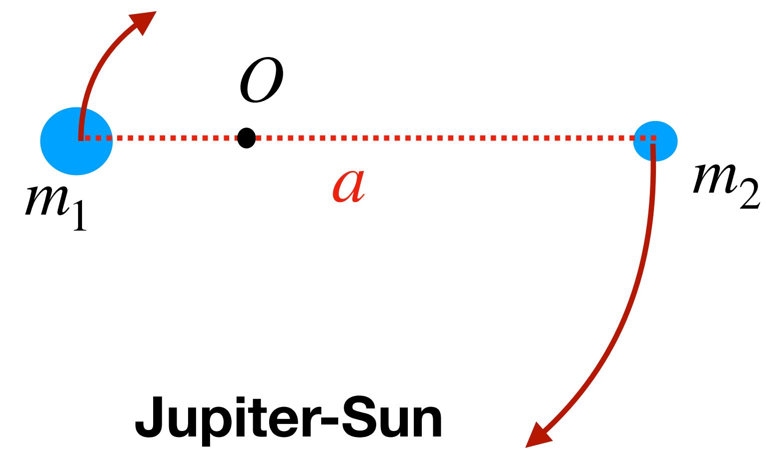

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„natural" unit of luminosity in GR

$3.63 \cdot 10^{52}\,\mathrm{W}$

Earth-Sun

$L \approx 200W$

Jupiter-Sun

$L \approx 5.2\,\mathrm{kW}$

Hulse-Taylor binary pulsar

$m_1 = 1.441\,M_\odot$ $\quad m_2 = 1.387\,M_\odot$

$a = 1.9 \cdot 10^6\,\mathrm{km}$ $\quad P = 7.75\,\mathrm{hr}$

~~$L = 6.6 \cdot 10^{23}\,\mathrm{W}$~~

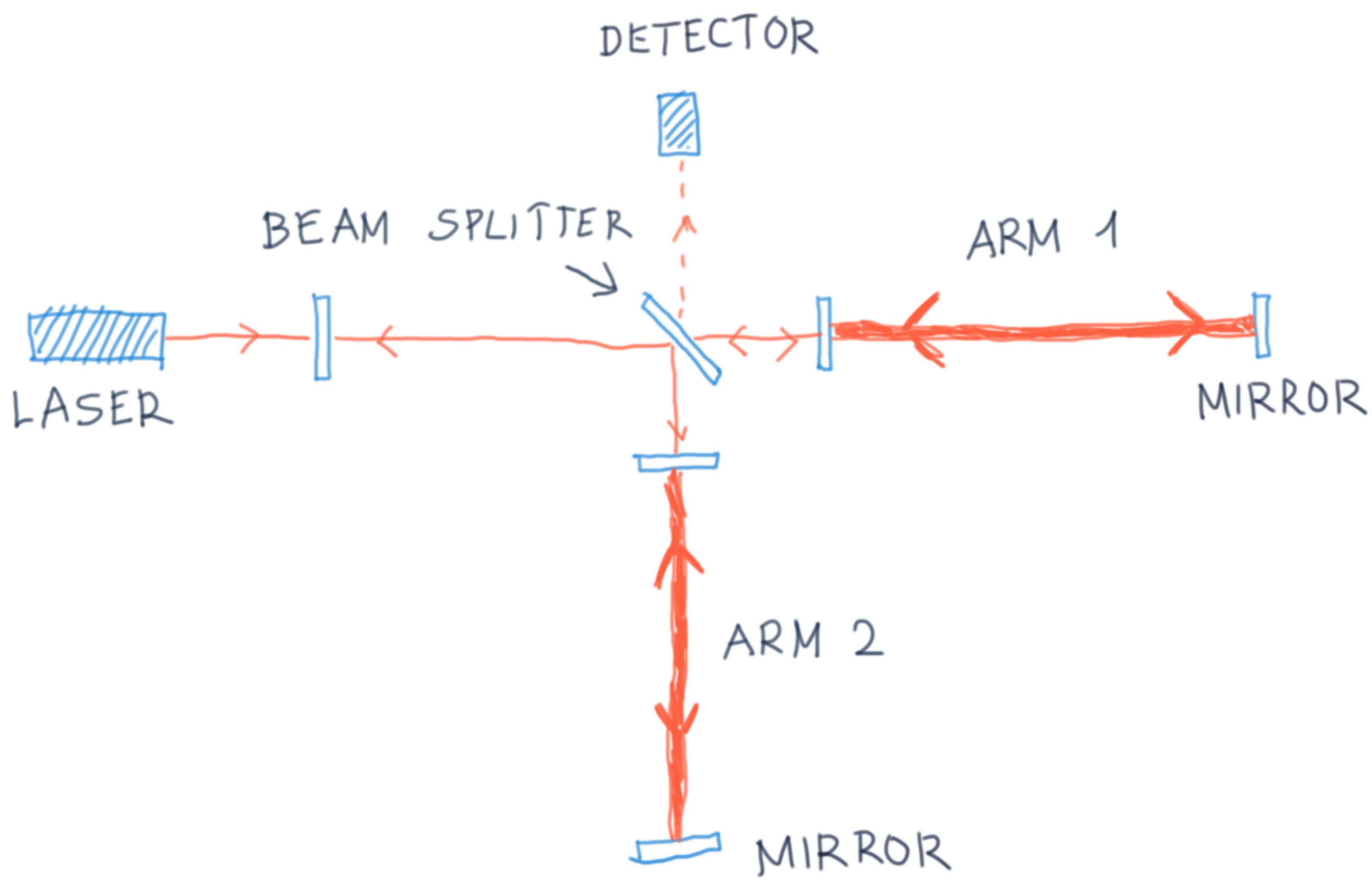
but $e = 0.617$, so $L \approx 7 \cdot 10^{24}\,\mathrm{W}$

leads to an observable decrease of the orbital period

$$\frac{dP}{dt} = -7.2 \cdot 10^{-15}\,\mathrm{s\,yr^{-1}}$$

Gravitational waves

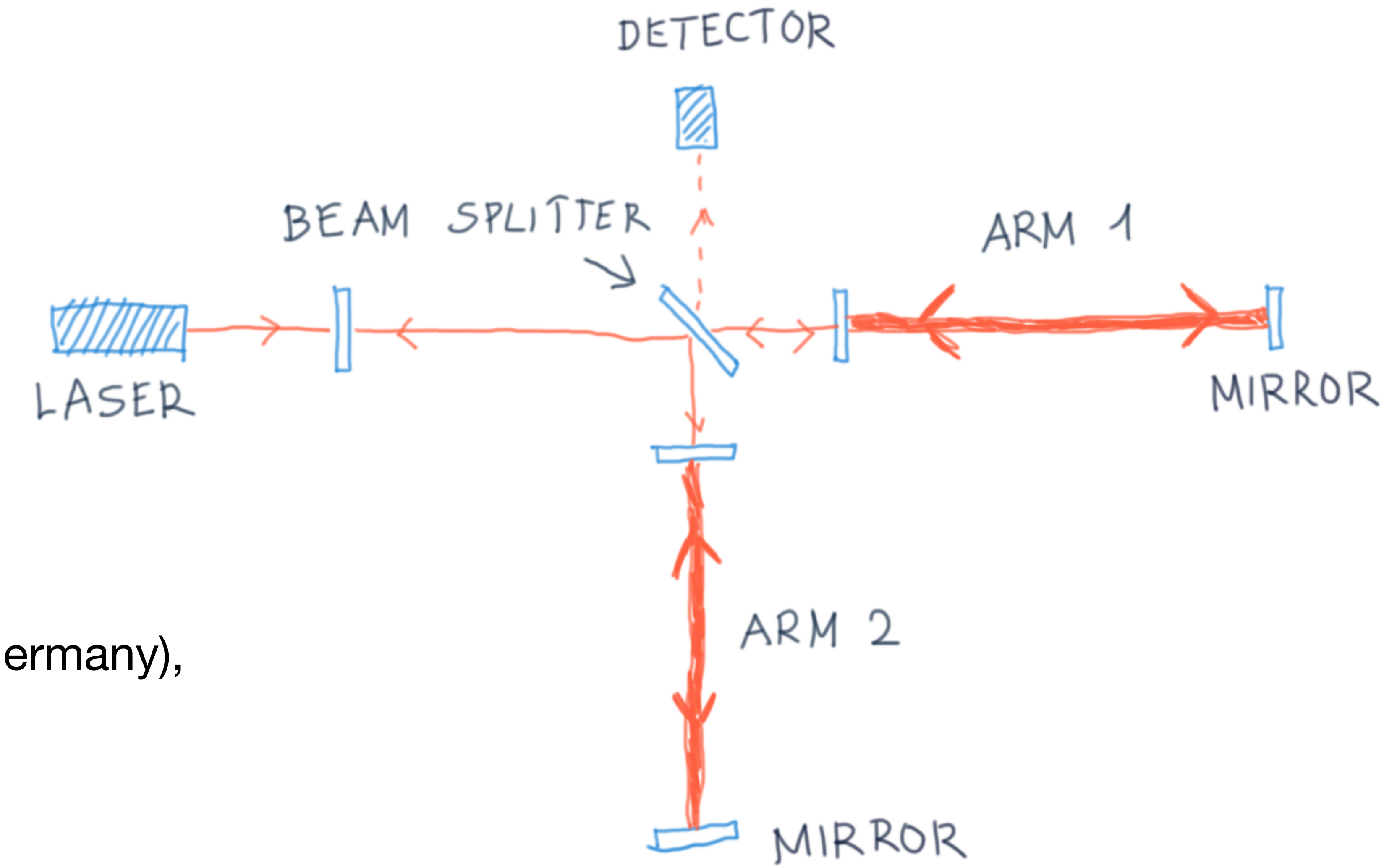
Detectors: interferometers



Earth-based detectors

Gravitational waves

Detectors: interferometers

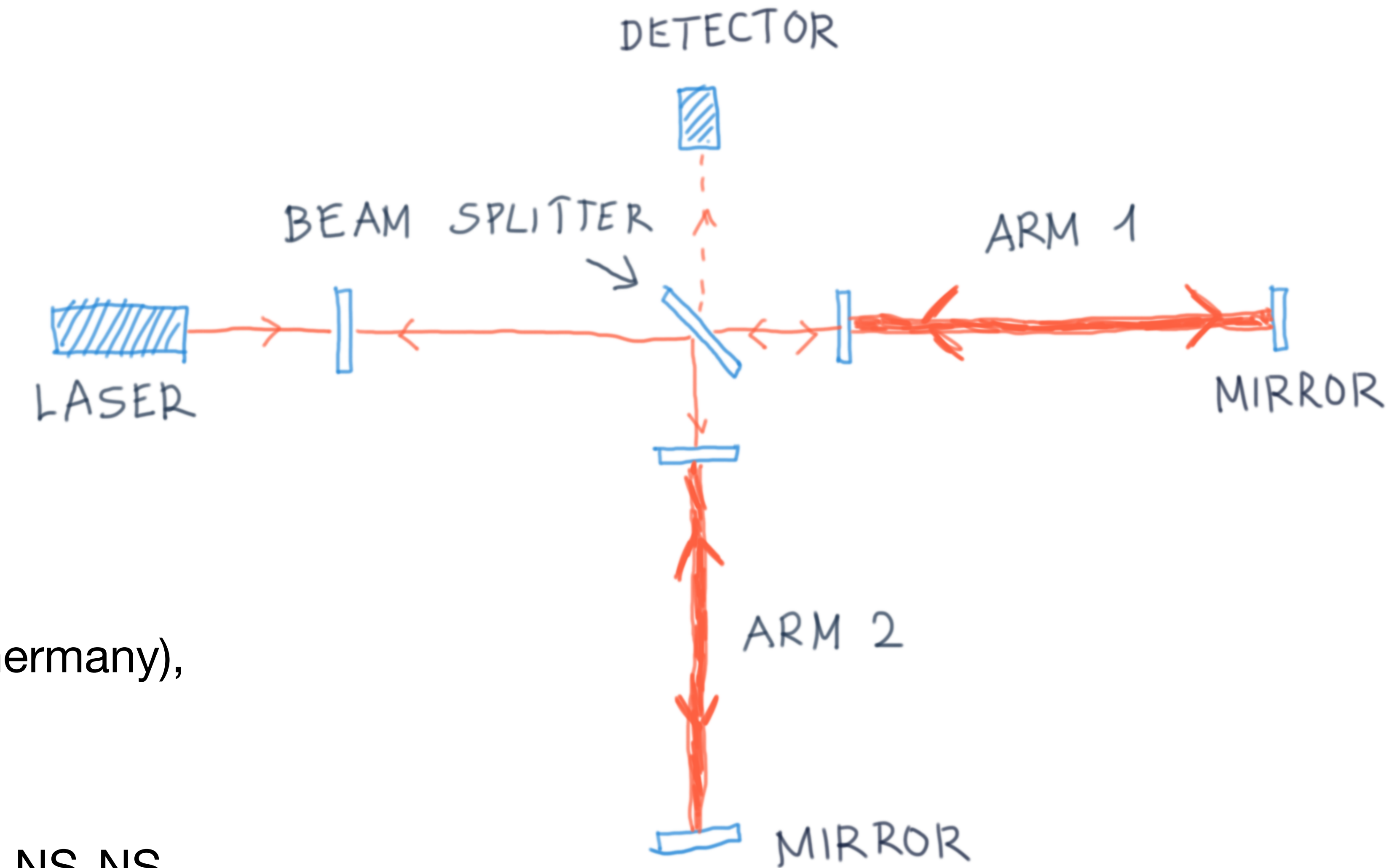


Earth-based detectors

LIGO (USA), VIRGO (Italy), GEO600 (Germany),
TAMA300 (Japan)

Gravitational waves

Detectors: interferometers



Earth-based detectors

LIGO (USA), VIRGO (Italy), GEO600 (Germany), TAMA300 (Japan)

Sources: coalescing compact binaries NS-NS, BH-NS, BH-BH systems

Gravitational waves

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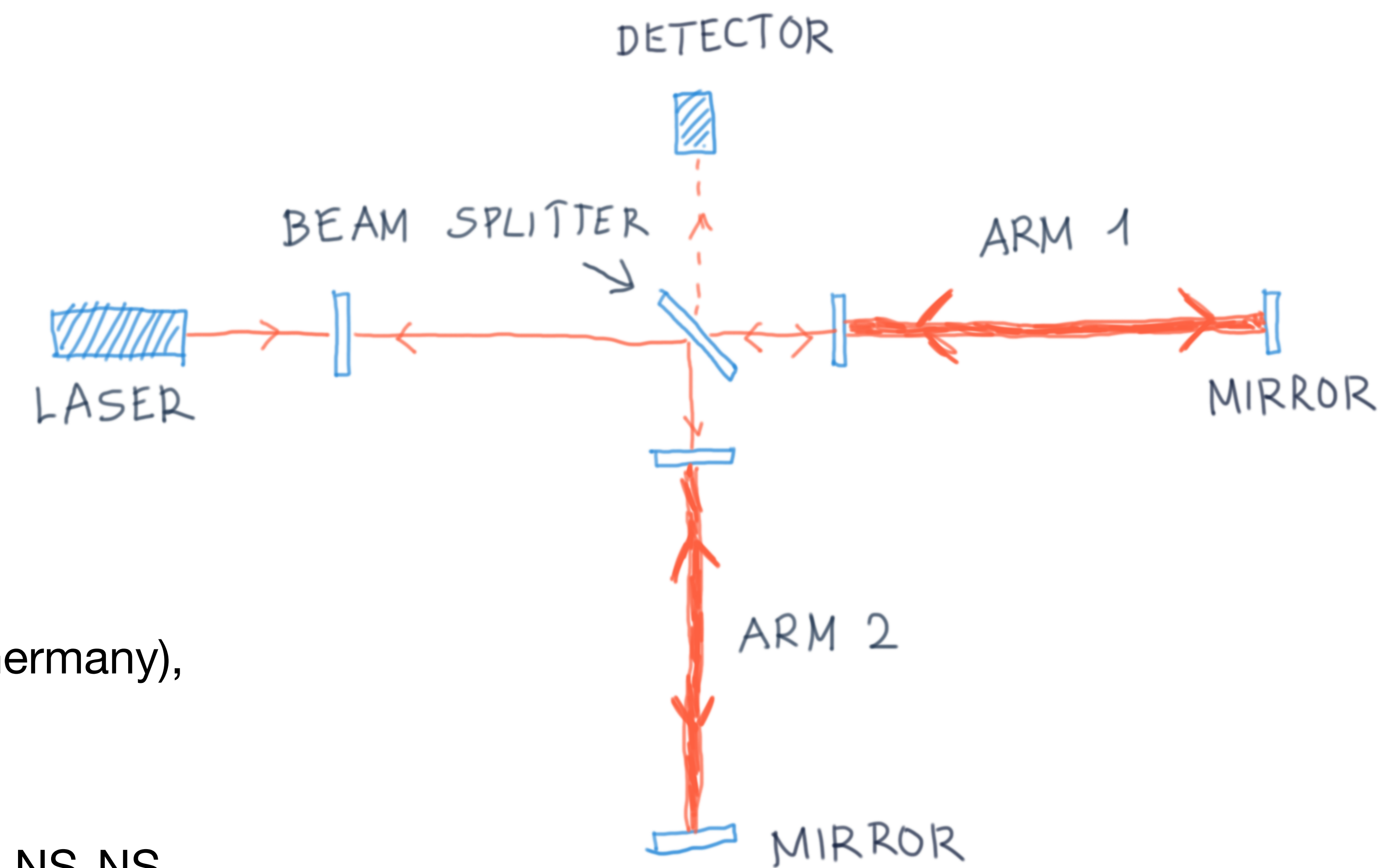

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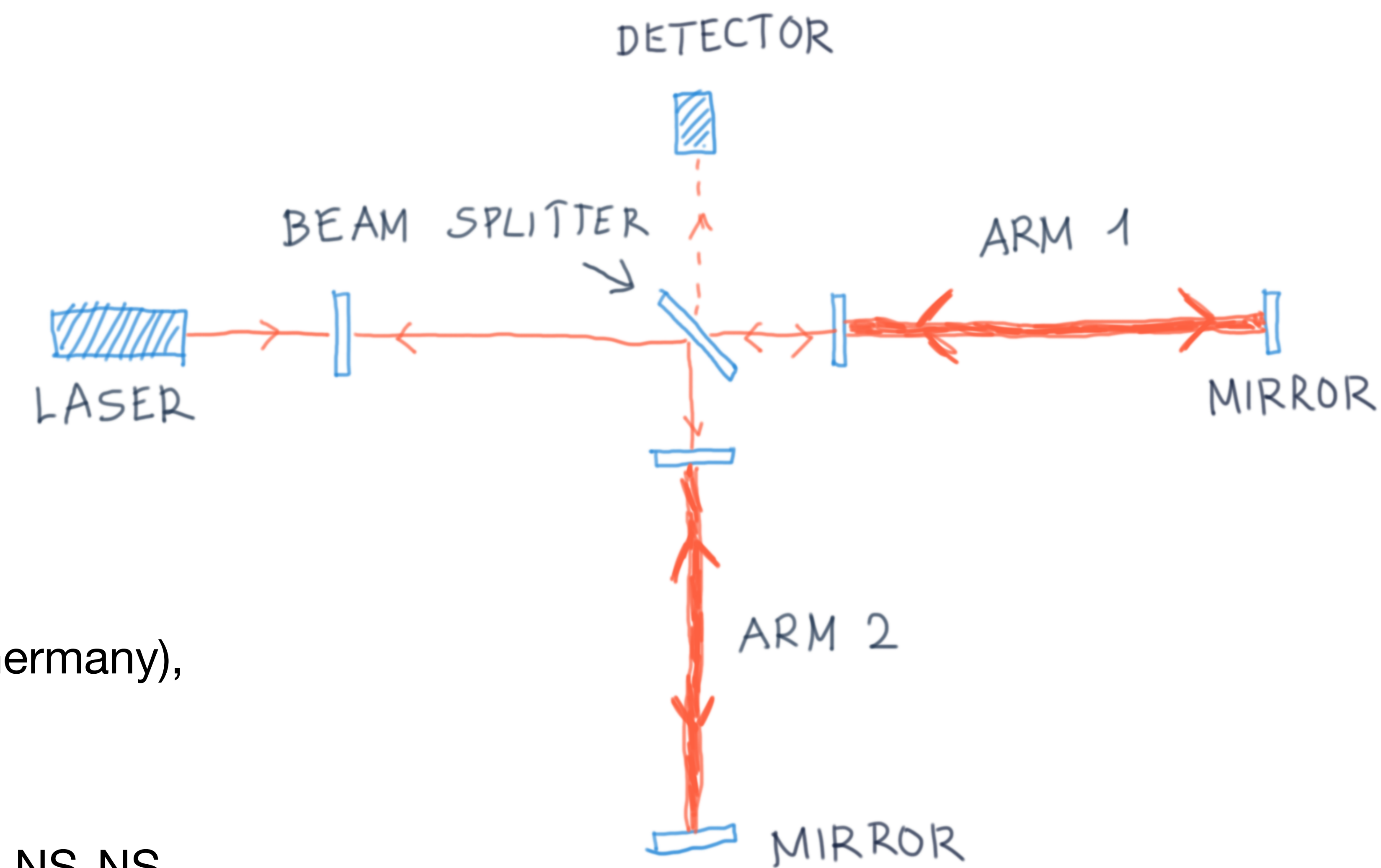

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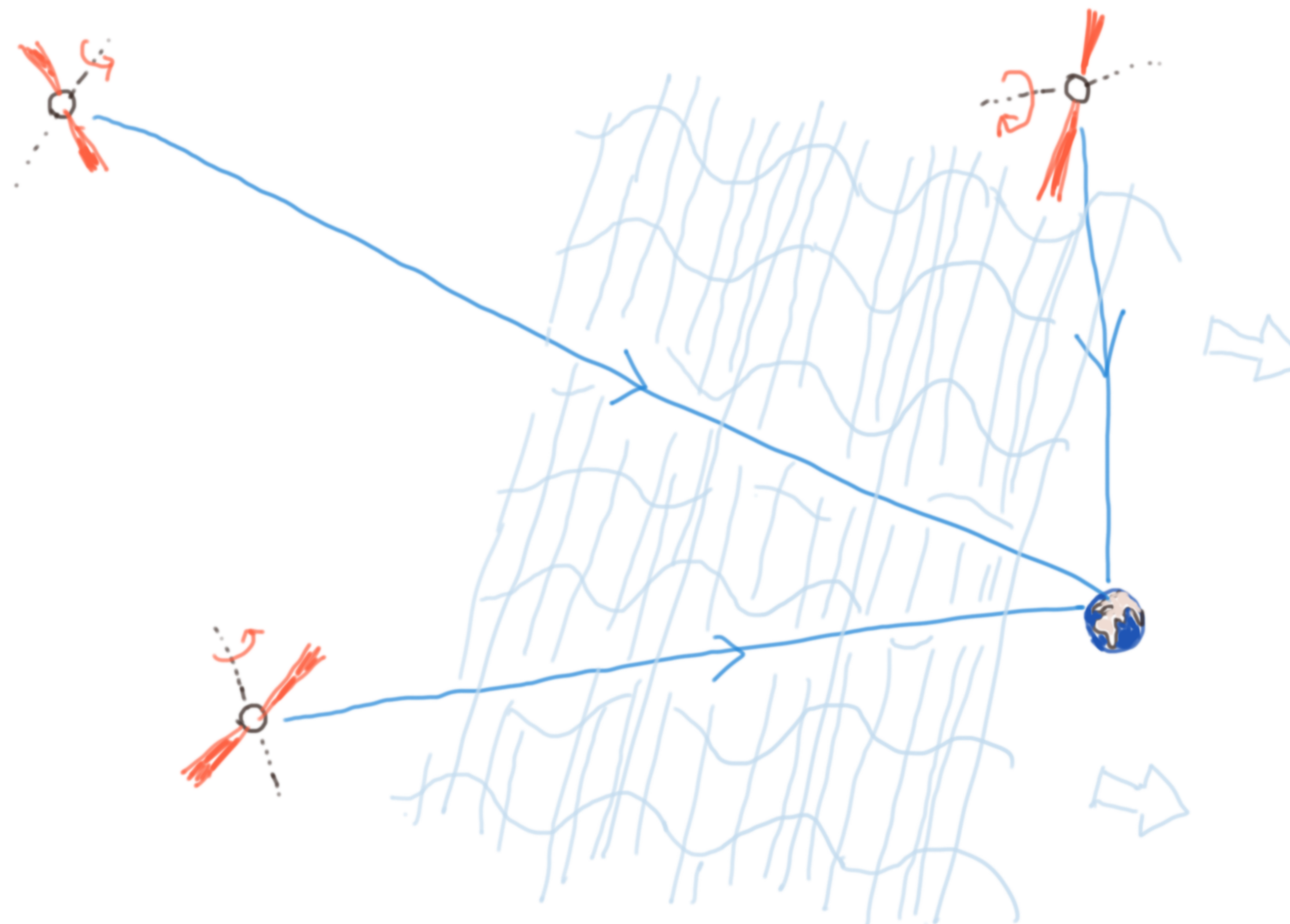
Many detections (around 100?) since the first one in 2015

Gravitational waves

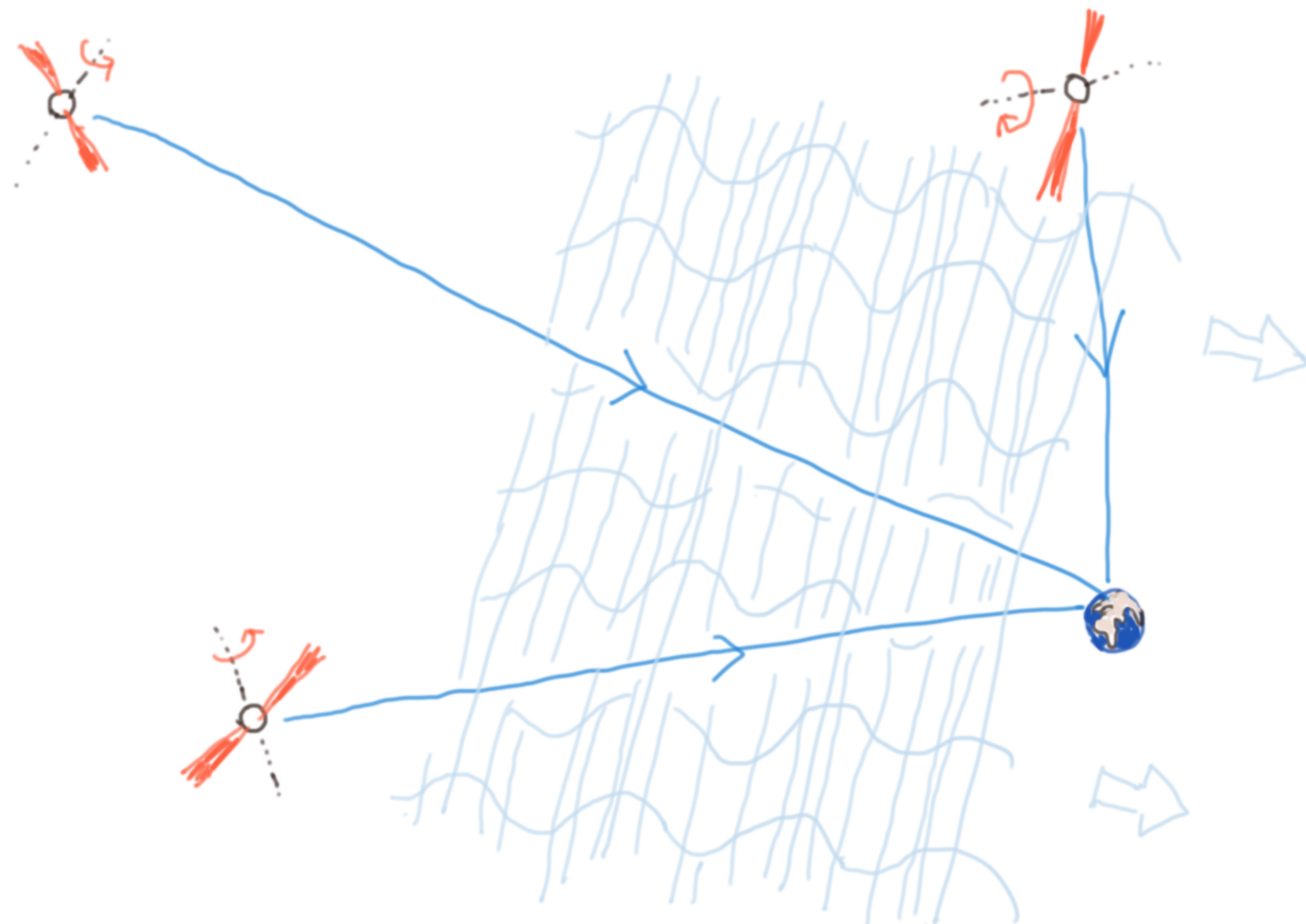
Detectors: pulsar timing arrays



Gravitational waves

Detectors: pulsar timing arrays

Similar physical principle: variations of TOA of electromagnetic waves



Gravitational waves

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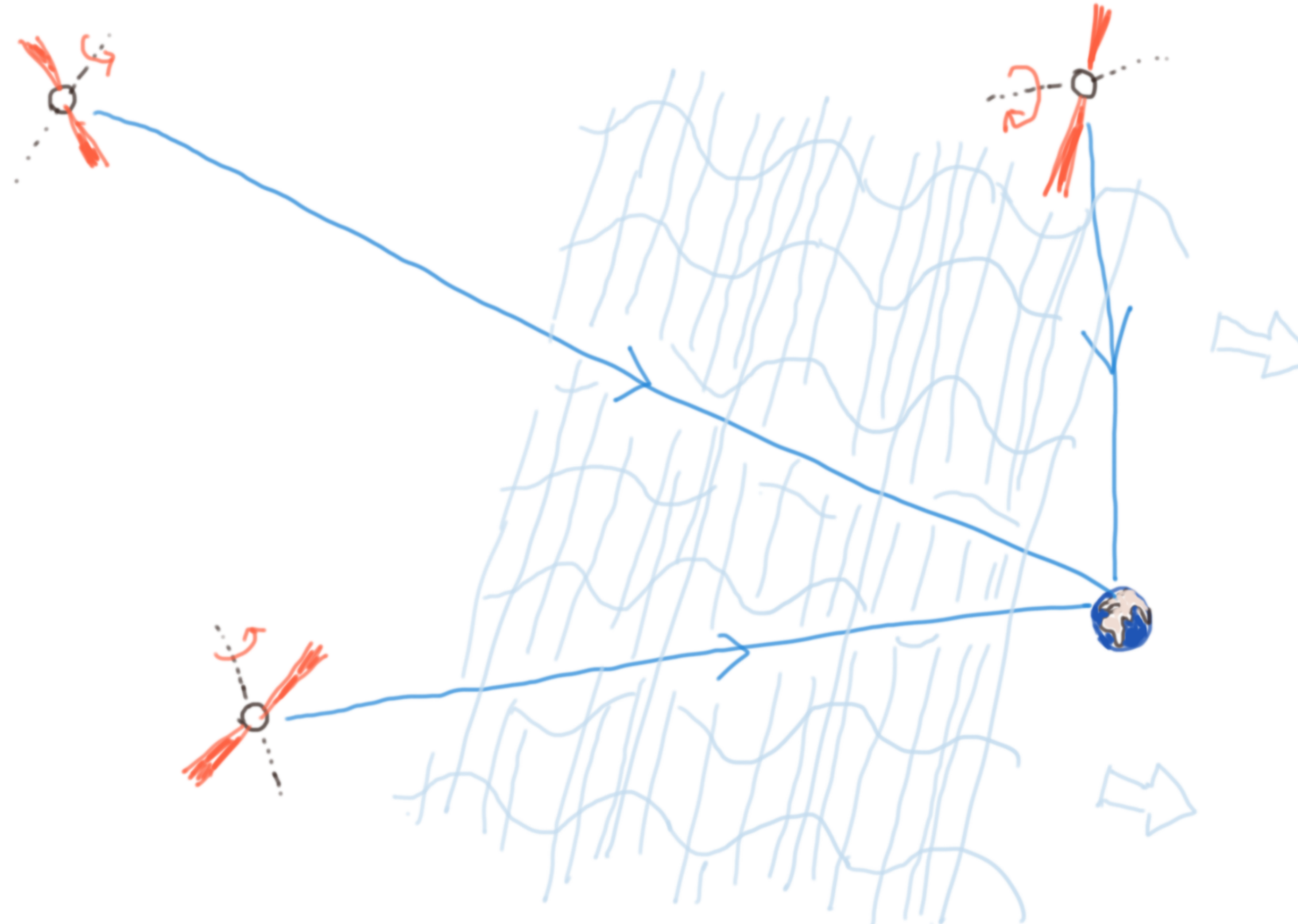
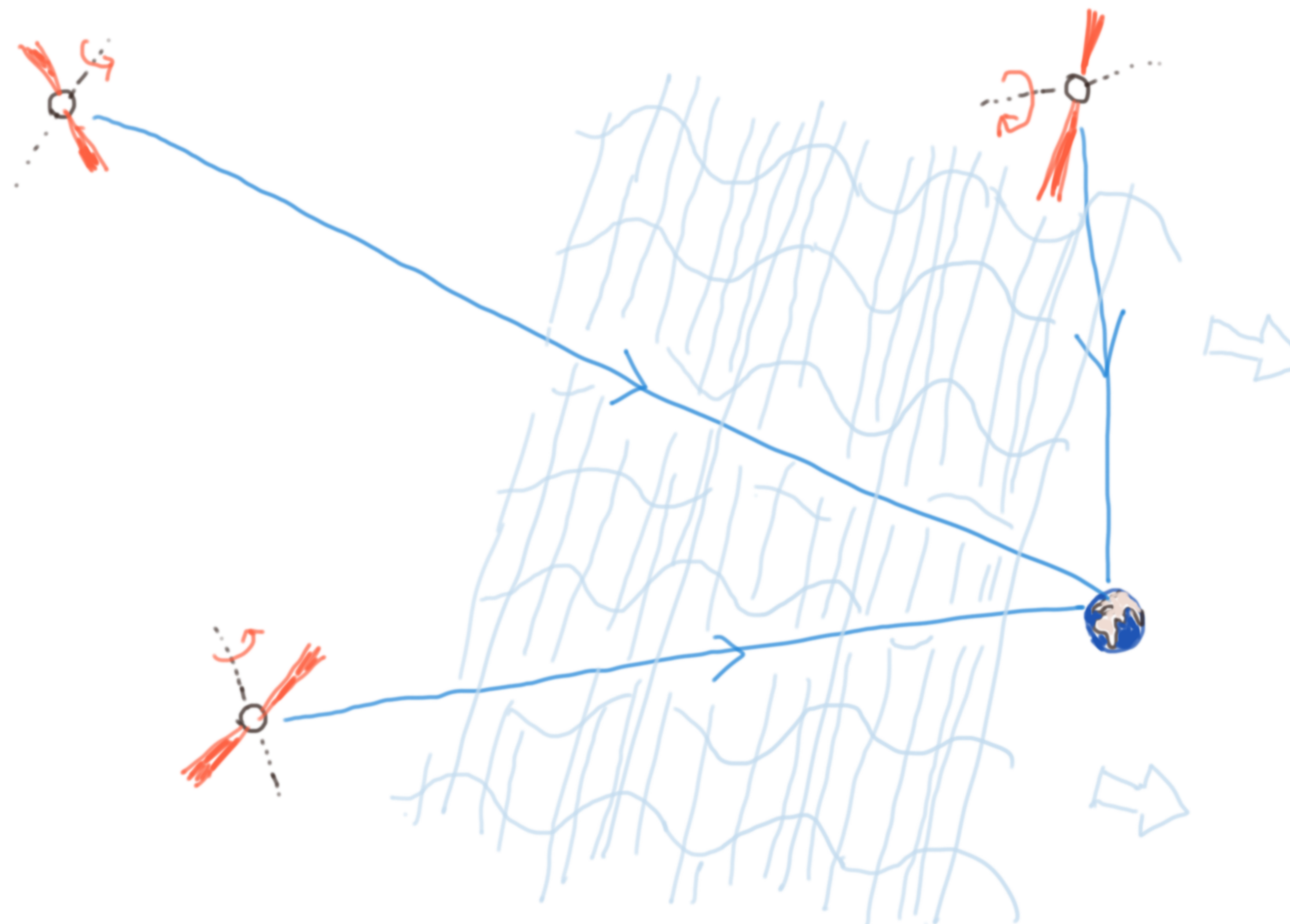


Similar physical principle: variations of TOA of electromagnetic waves

Distances much larger than wavelengths

Gravitational waves

Detectors: pulsar timing arrays



Similar physical principle: variations of TOA of electromagnetic waves

Distances much larger than wavelengths

GW stochastic background = additional source of noise in TOA's,

Gravitational waves

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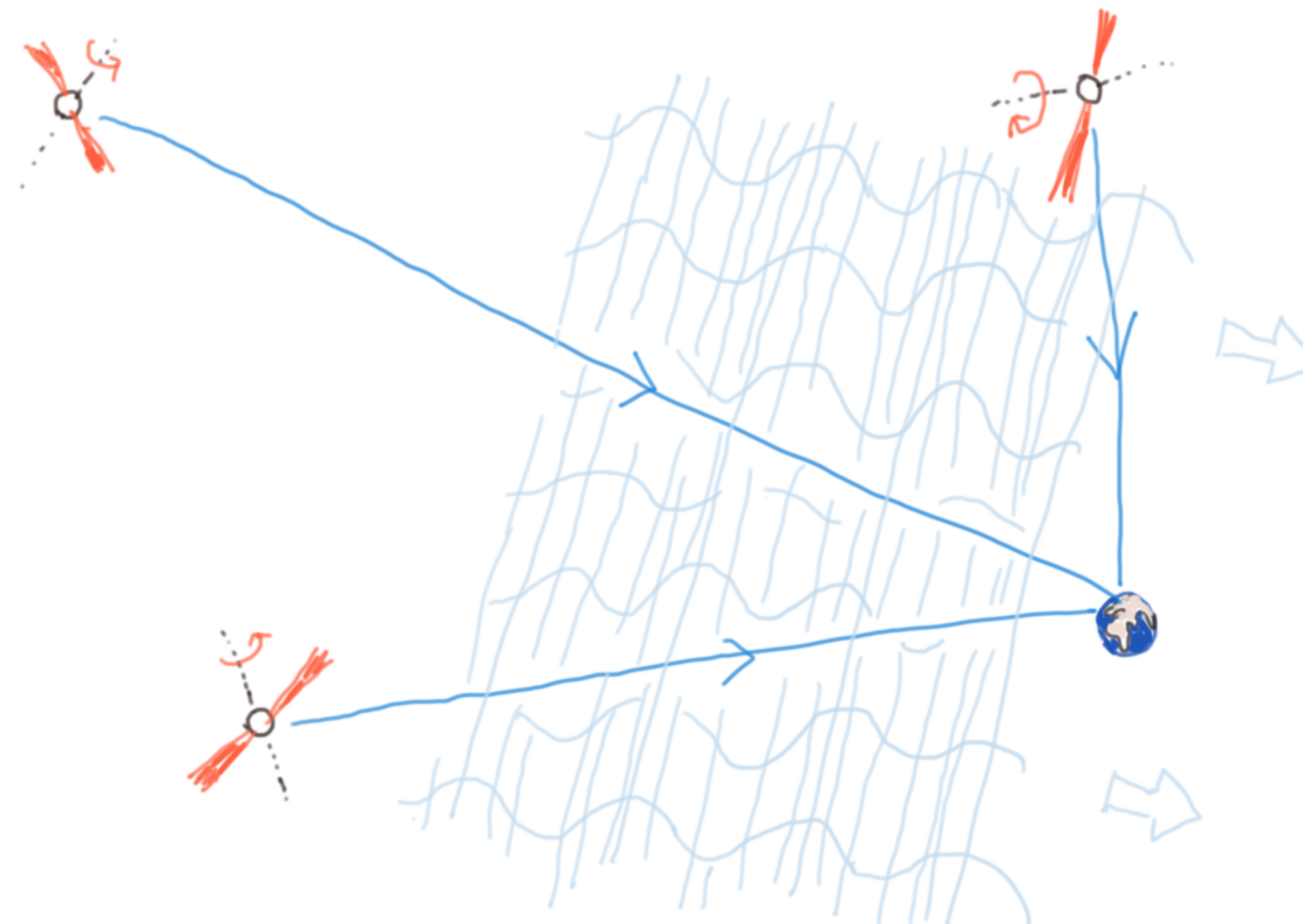

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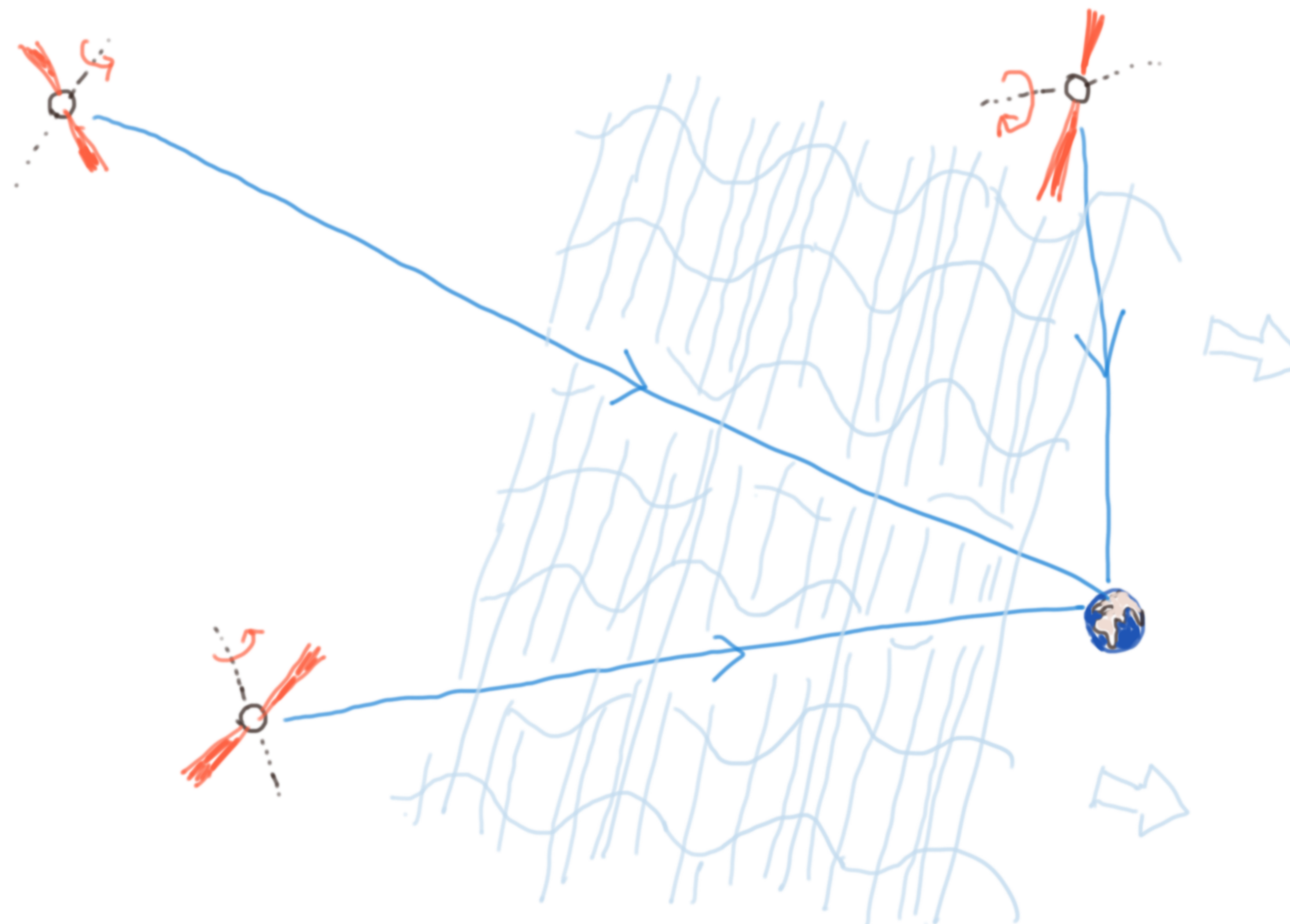

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Evidence for signal (2023): NANOGrav, EPTA, PPTA, InPTA, ChPTA.
Frequency band: $1 - 10\,\mathrm{nHz} \approx 0.1 - 1\,\mathrm{yr}^{-1}$

Gravitational waves

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End of Lecture 9

Schwarzschild solution

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Assumptions

Spherically symmetric

Static = stationary (not changing with time t) + time reversal symmetry $t \to -t$

Vacuum, i.e. $R_{\mu\nu} = 0$

Schwarzschild solution

Spherically symmetric + static

Exact definitions: see

- Misner, Thorne, Wheeler „*Gravitation*" Ch. 23
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stationary (time-independent) spacetime made of such 3-spaces

$$g = g_{tt}(r)\, dt^2 + g_{rr}(r)\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + 2g_{tr}(r)\, dt\, dr$$

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$$g = -e^{2\alpha(r)}\,dt^2 + e^{2\beta(r)}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$$

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Combine R_{tt} and R_{rr}

$$0 = e^{2(\beta(r)-\alpha(r))}\,R_{tt} + R_{rr} = \frac{2}{r}\,\big(\alpha'(r) + \beta'(r)\big) \qquad \Longrightarrow \alpha(r) = -\beta(r) + C$$

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$$t \to \tilde{t} = e^{C}\, t$$

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$$e^{2\alpha(r)} = 1 - \frac{R_s}{r}$$

R_S (Schwarzschild radius) is a constant of the dimension of length

Schwarzschild solution

Schwarzschild metric

$$g = -\left(1 - \frac{R_S}{r}\right) dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

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- 4 independent Killing vectors

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- For $r \to \infty$ we have $g \to g_0 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$

Minkowski metric in spherical coordinates

Far away from the center the metric appears to be very close to a flat one

asymptotic flatness