Introduction to general relativity

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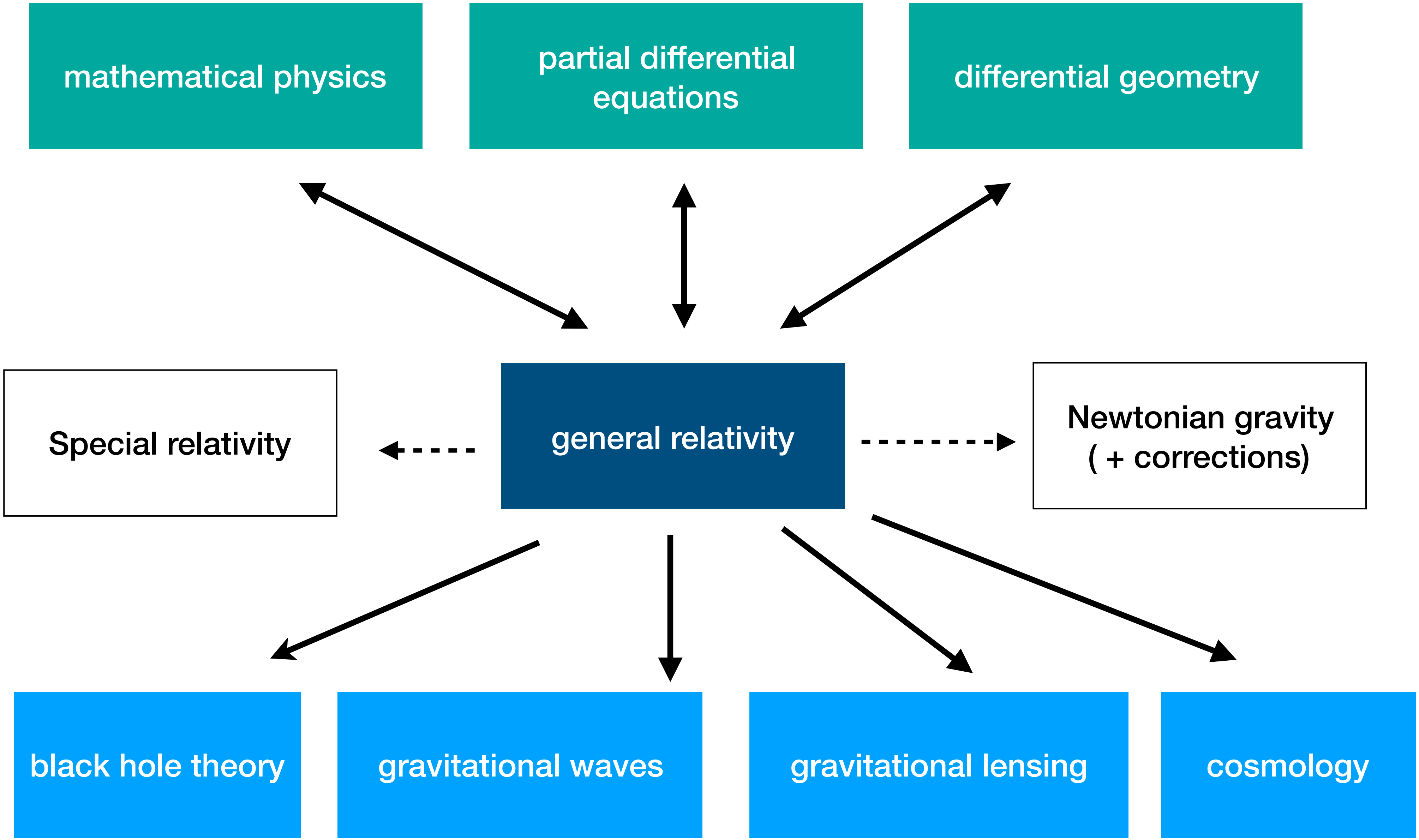
General relativity - intro

- Current theory of gravitational forces and the geometry of spacetime
- Underlies much of astrophysics and cosmology
- Purely classical theory, no quantum effects
- Developed mostly by A. Einstein 1907-1915, influences from M. Grossmann, D. Hilbert

General relativity - intro

- Gravity = weakest of all 4 fundamental interactions (electromagnetic, strong, weak, gravitational).
- For 2 protons: $\frac{F_{grav}}{F_{EM}} = \frac{G\, m_P^2/r^2}{e^2/(4\pi\epsilon_0 r^2)} \approx 10^{-36}$
- It has no preferred length/energy scale
- Becomes significant only if we accumulate large masses
- But: long range interaction, cannot be screened/shielded (unlike EM forces).
- Therefore: dominating force on largest scales

General relativity - intro



History of GR

1907-1916 A. Einstein (D. Hilbert, M. Grossman) - development of GR

1916 A. Einstein - gravitational waves in linearized (weak) gravity. But are they real?

1917 K. Schwarzschild - first solution: massive body, black hole

1919 A. Eddington - measurement of light ray bending by the Sun

1922-1935 A. Friedmann, G. Lemaître, H. P. Robertson, A. Walker - expanding Universe solutions. Beginning of modern (theoretical) cosmology

1920-1950's Geometry of the Schwarzschild black hole (A. Eddington, G. Lemaître, D. Finkelstein, M. Kruskal), event horizon

1920-1930's A. Einstein, A. Eddington, R. Mandl… - masses may act as lenses for light

1920's-1960's Exact GW solutions - Brinkmann, A. Einstein and N. Rosen, J. Ehlers and W. Kundt, I. Robinson and A. Trautman

1960's A. Trautman, H. Bondi, F. Pirani, I. Robinson - GW are real (carry energy)

1959 Einstein field equations formulated as evolution equations (R. Arnowitt, S. Deser, C. Misner)

History of GR

1960's other tests of GR in Solar System and on Earth: R. Pound and G. Rebka, I. Shapiro, R. Dicke…

1963 R. Kerr - spinning black hole solution (E. T. Newman, J. N. Goldberg, R. K. Sachs, …)

1960's-1970's R. Penrose, S. Hawking - singularity theorems (black holes, Big Bang)

1970's Evidence for Cygnus X-1 being an accreting black hole

1971-78 double pulsar PSR 1913+16 and effects of its GW emission: R. A. Hulse, J. H. Taylor

1970's-1990's evidence for SMBH in the centers of many galaxies

1979 first strong gravitational lens, double image of a quasar (Walsh, Cardswell, Waymann)

1960's-1980's gravitational lensing theory (S. Refsdahl, R. Bourassa, R. Kantowski, P. Schneider, B. Paczyński…)

1990's-2010's Numerical relativity (M. Choptuik, …), binary black hole mergers and their GW emission (F. Pretorius, …)

2015 First detection of gravitational wave using interferometers (LIGO, Virgo)

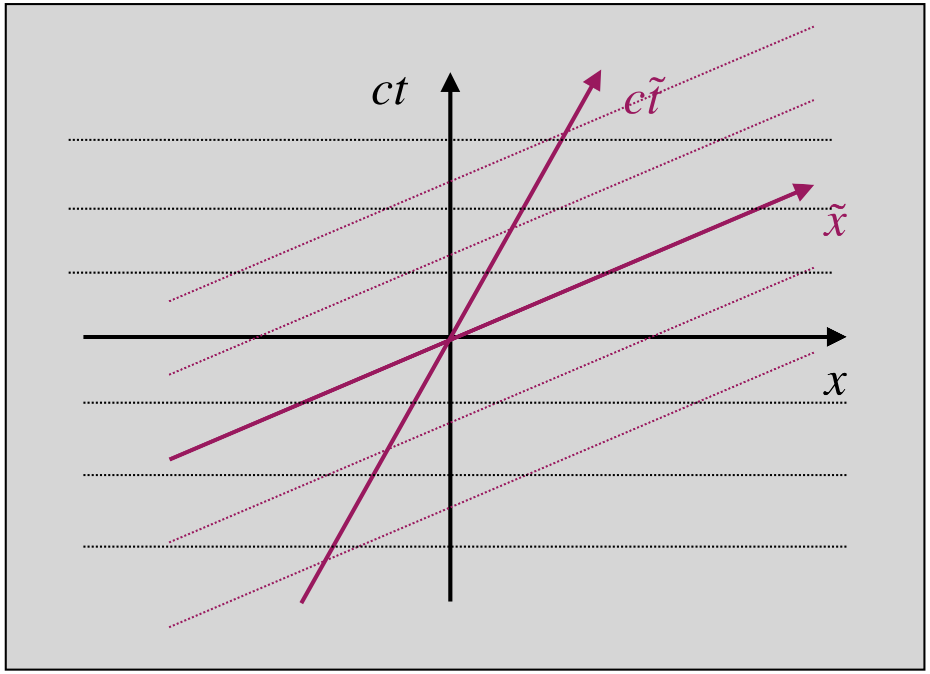
2017 Black hole shadow image of M87 (Event Horizon Telescope)

History of GR

Nobel Prizes in Physics related to GR (or distantly related)

- **2020** Roger Penrose *"for the discovery that black hole formation is a robust prediction of the general theory of relativity"*, Reinhard Genzel and Andrea Ghez *"for the discovery of a supermassive compact object at the centre of our galaxy"* (black holes)
- **2019** James Peebles *"for theoretical discoveries in physical cosmology"* (cosmology)
- **2017** Rainer Weiss, Barry C. Barish and Kip S. Thorne *"for decisive contributions to the LIGO detector and the observation of gravitational waves"* (gravitational waves)
- **2011** Saul Perlmutter, Brian P. Schmidt and Adam G. Riess *"for the discovery of the accelerating expansion of the Universe through observations of distant supernovae"* (cosmology)
- **2006** John C. Mather and George F. Smoot *"for their discovery of the blackbody form and anisotropy of the cosmic microwave background radiation"* (cosmology)
- **1993** Russell A. Hulse and Joseph H. Taylor Jr. *"for the discovery of a new type of pulsar, a discovery that has opened up new possibilities for the study of gravitation"* (gravitational waves)
- **1983** Subramanyan Chandrasekhar *"for his theoretical studies of the physical processes of importance to the structure and evolution of the stars"*

Special relativity - brief summary

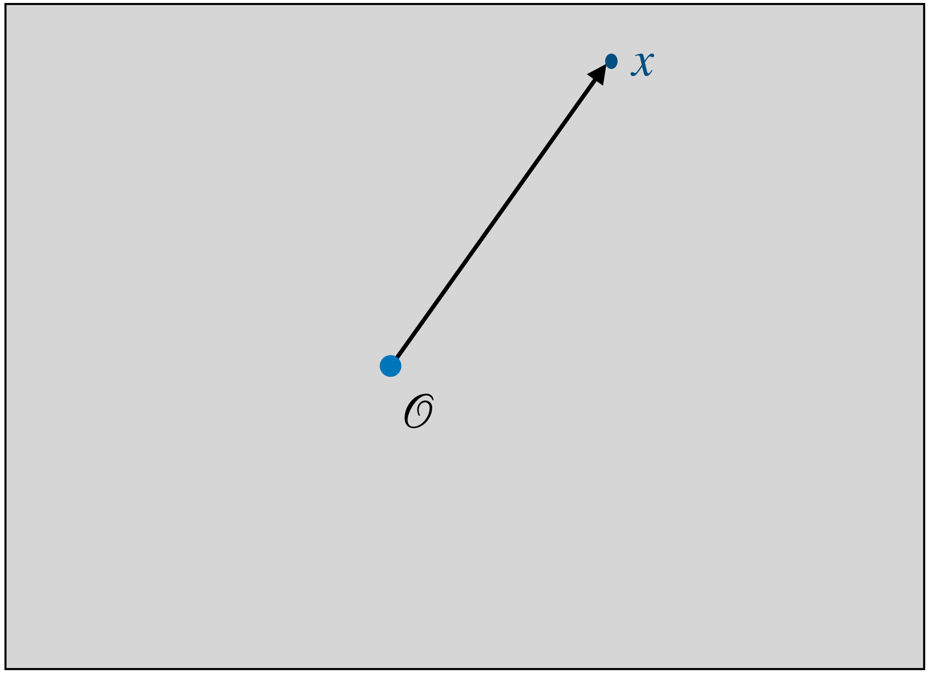


Just a summary, see:

- B. Schutz, **"A First Course in General Relativity"**
- J. B. Hartle, **„Gravity: An Introduction to Einstein's General Relativity"**
- **...**

Special relativity

- Physical phenomena take place in **spacetime** (4D object)
- Affine space with 3+1 dimensions
- points = **events**



indices, not exponents

Special relativity

- Physical phenomena take place in **spacetime** (4D object)

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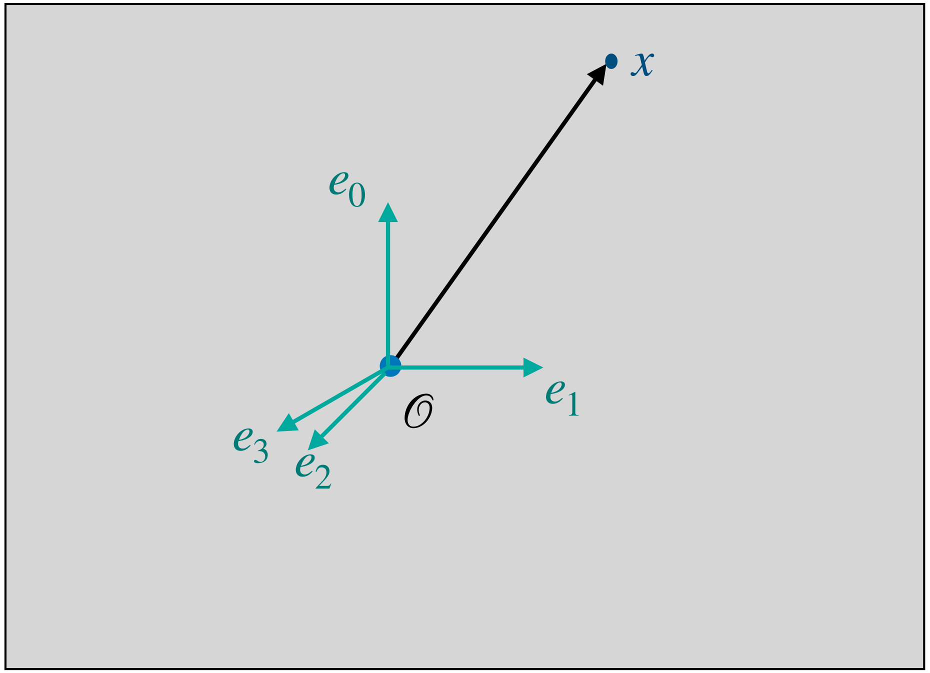
- points = **events**

- **Inertial frame** = idealized system of synchronized clocks and ranging/measuring devices used for assigning coordinates to points



- mathematical definition: basis of 4 vectors, defining the coordinate system

$$x - \mathcal{O} = x^0 e_0 + x^1 e_1 + x^2 e_2 + x^3 e_3 \qquad (ct, x, y, z) = \left(x^0, x^1, x^2, x^3\right)$$

indices, not exponents

Special relativity

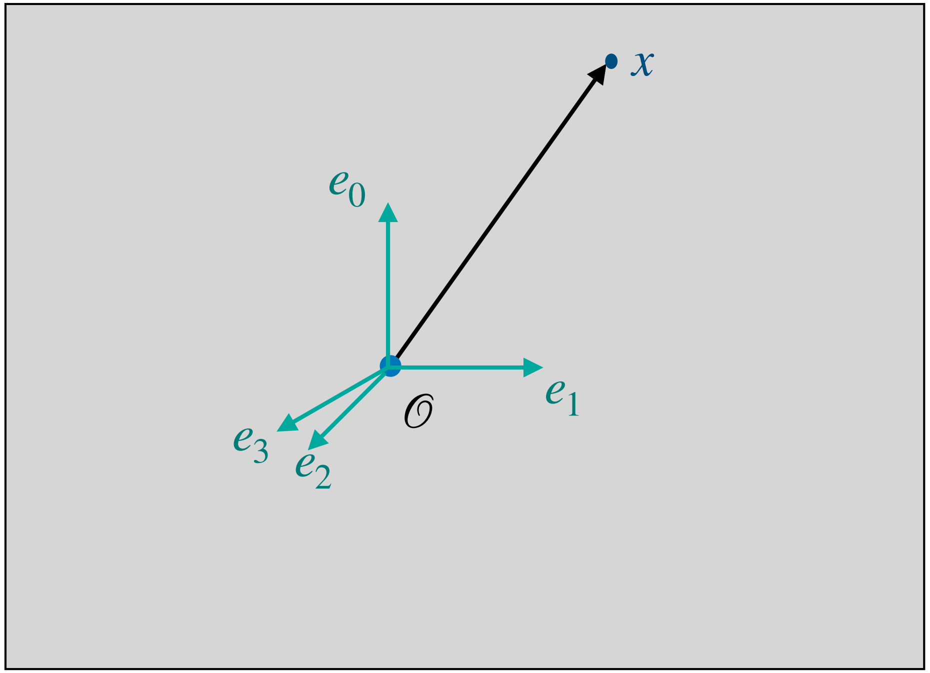
- Intuition: e_0 corresponds to time flow (motion of an observer related to that frame)

- e_1, e_2, e_3 define purely spatial directions, orthonormal. Span a 3-space

- geometry of 3-space = Euclidean

- frame usually connected to a physical observer (body)

$$x^1 = x^2 = x^2 = 0$$



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Special relativity - postulates

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    - ***Principle of relativity*** - inertial frames move with respect to each other with constant velocity. They are equivalent from the point of view of mechanics (and other physical laws)

      identical experiments $\Longrightarrow$ the same results in any inertial frame

      cannot detect absolute velocity of an inertial frame without an external reference (no absolute motion)

      laws of physics take a simple form in inertial frames

      assumed also in Newtonian mechanics

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  - ***Speed of light in vacuum is constant*** - speed of light in vacuum is exactly $c = 299\,792\,458\,\mathrm{m/s}$ as measured in any inertial frame

    we need to abandon:

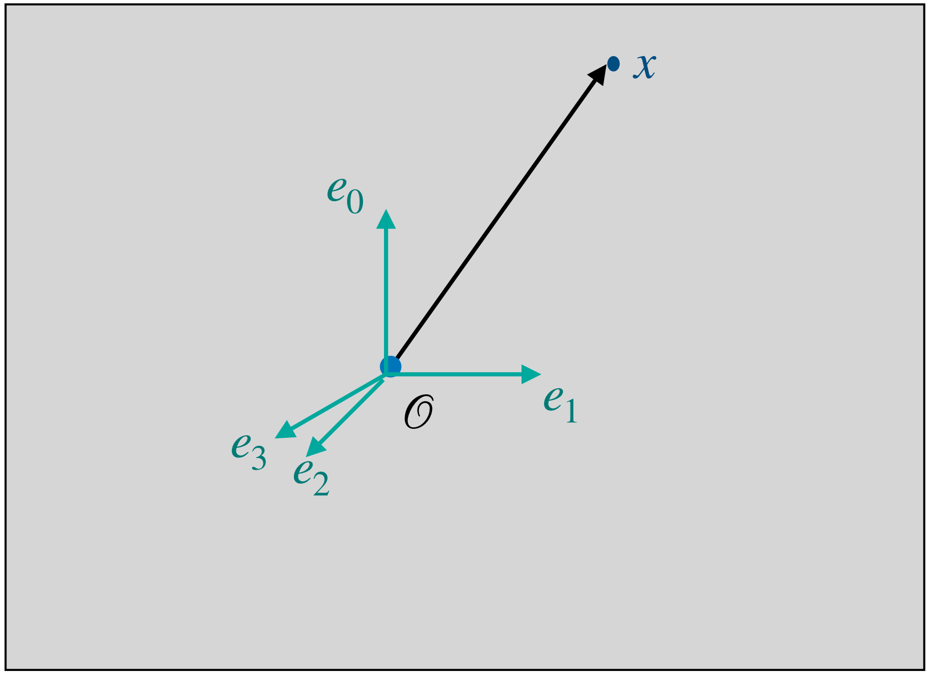
    standard notion of global, absolute time (relativity of simultaneity, time dilation)

    notion of frame-independent distances (Lorentz contraction)

    additivity of velocities

Special relativity

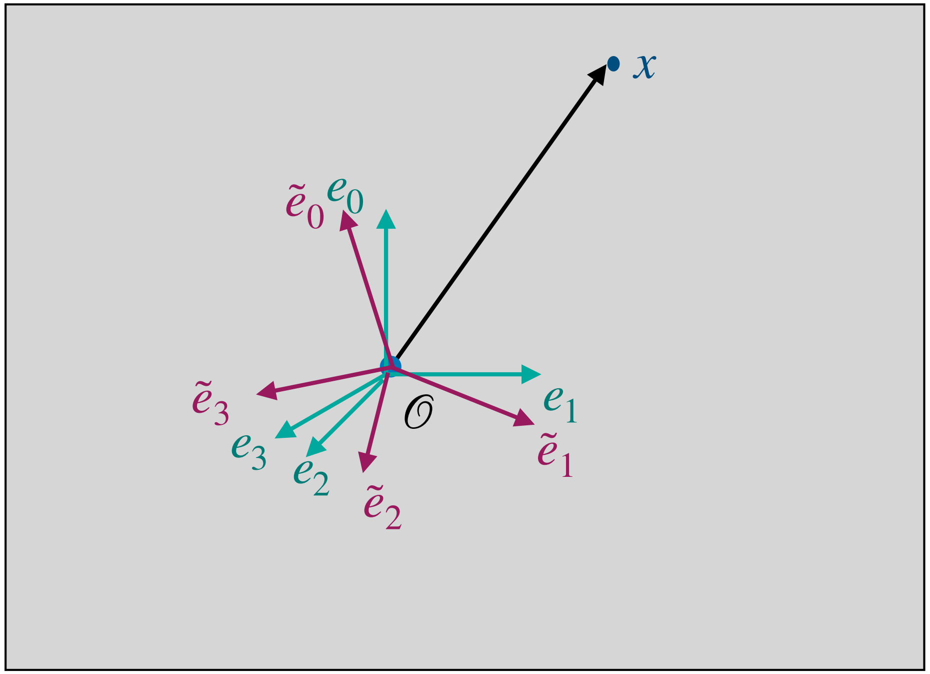
Inertial frames related to each other by **Lorentz transformations**



Special relativity

Inertial frames related to each other by
Lorentz transformations

$$\Lambda^T \eta \Lambda = \eta \qquad \eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



$$\tilde{e}_\nu = \sum_{\mu=0}^{3} \left(\Lambda^{-1}\right)^\mu{}_\nu \, e_\mu$$

$$\tilde{x}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu \, x^\nu \quad + b^\mu$$

if we shift the origin

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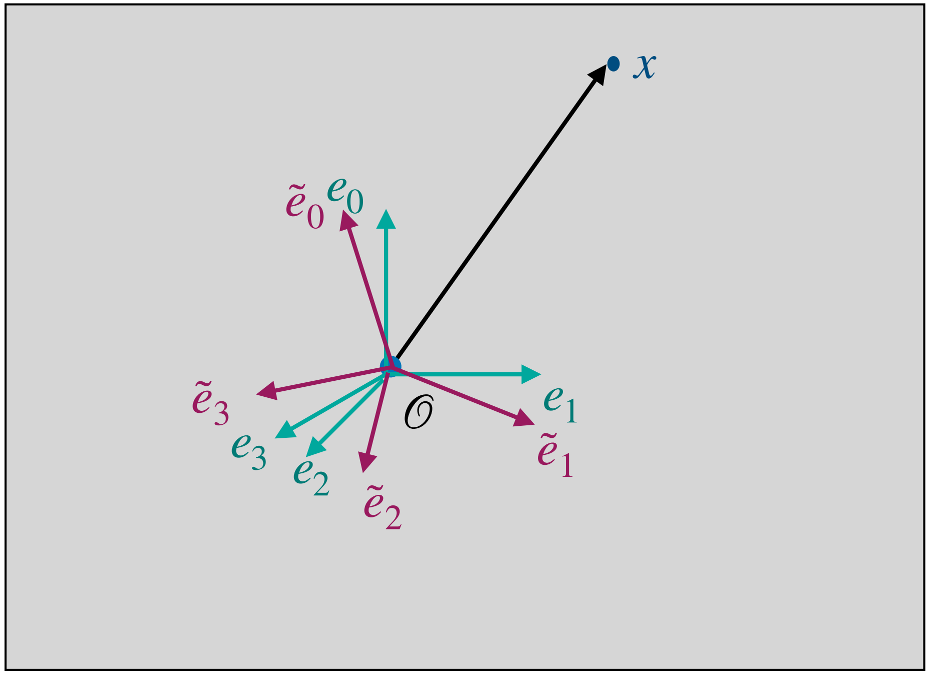

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transforms in general mix time and space dimensions

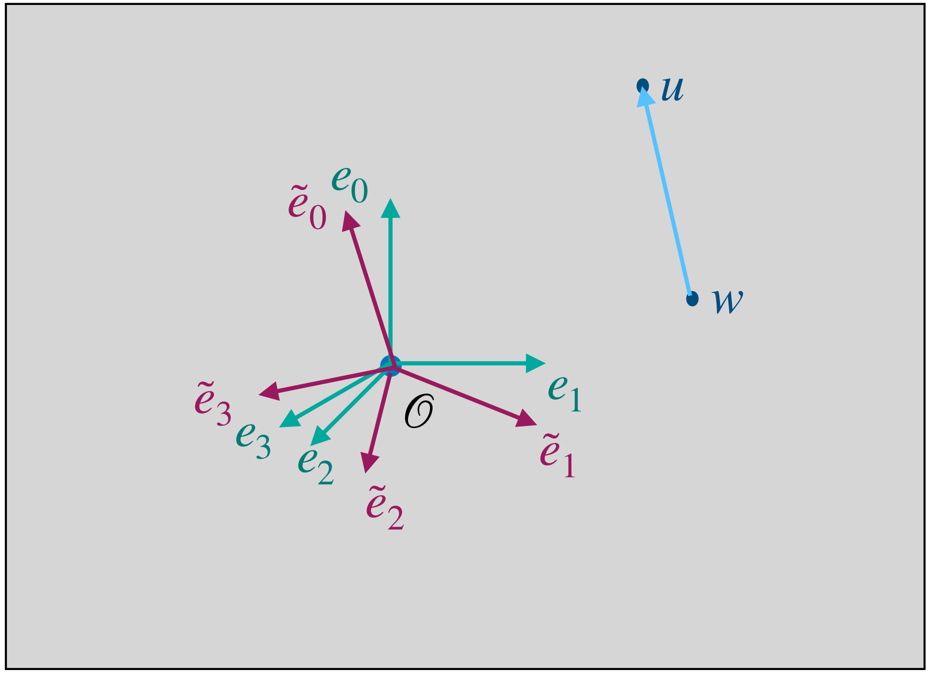
Special relativity

- Invariant **interval** between events

$$u - w = \left(\Delta x^0\right) e_0 + \left(\Delta x^1\right) e_1 + \left(\Delta x^2\right) e_2 + \left(\Delta x^3\right) e_3$$

$$\Delta s^2 = -\left(\Delta x^0\right)^2 + \left(\Delta x^1\right)^2 + \left(\Delta x^2\right)^2 + \left(\Delta x^3\right)^2$$

$$\Delta s^2 = -\left(\Delta \tilde{x}^0\right)^2 + \left(\Delta \tilde{x}^1\right)^2 + \left(\Delta \tilde{x}^2\right)^2 + \left(\Delta \tilde{x}^3\right)^2$$



We may also define an invariant product of vectors

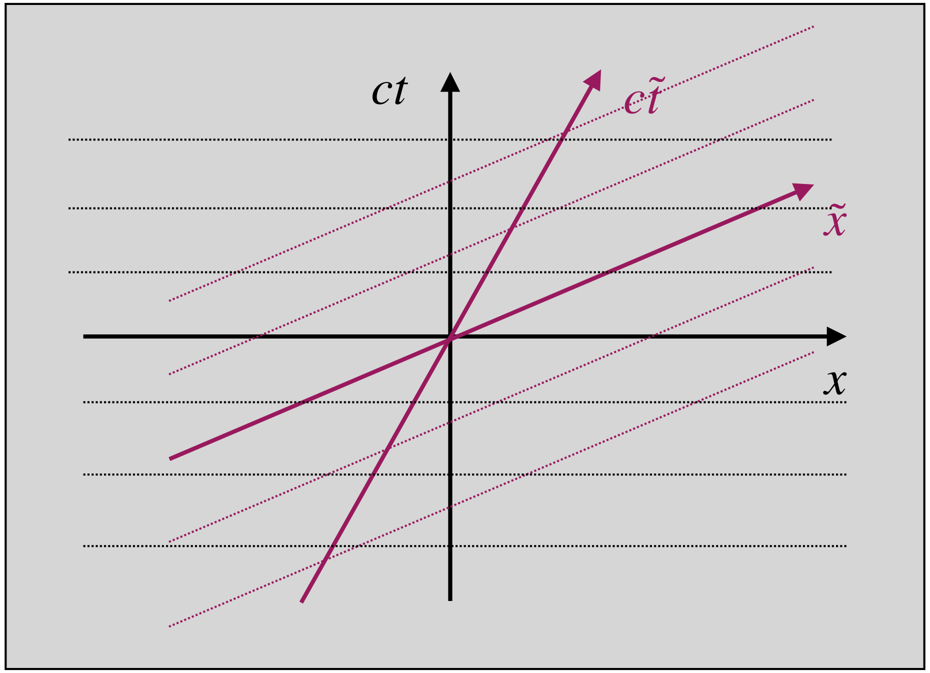
$$X \cdot Y = -X^0Y^0 + X^1Y^1 + X^2Y^2 + X^3Y^3 = \sum_{\mu,\nu} X^\mu Y^\nu \eta_{\mu\nu}$$

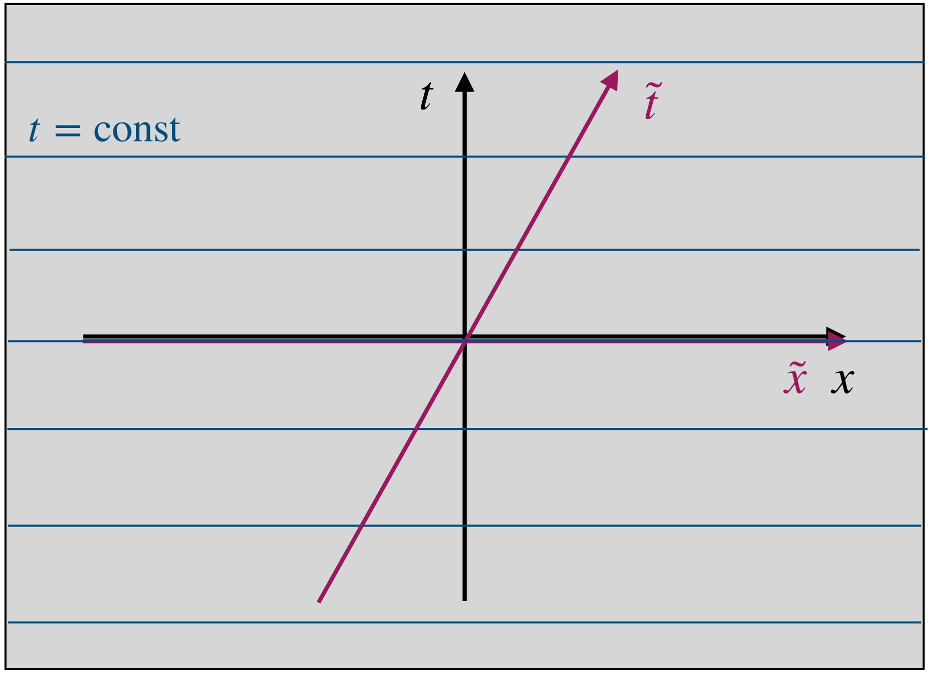
$$\Delta s^2 = (u - w) \cdot (u - w)$$

NOT positive definite, for two different points u, w Δs^2 may be > 0, < 0 and =0

SR vs. Newtonian physics

Inertial frame (observer) $\mathcal{O}$ and $\tilde{\mathcal{O}}$, moving with velocity v in direction x wrt $\mathcal{O}$





$$c\tilde{t} = \gamma\, ct - \gamma\, \frac{v\, x}{c}$$

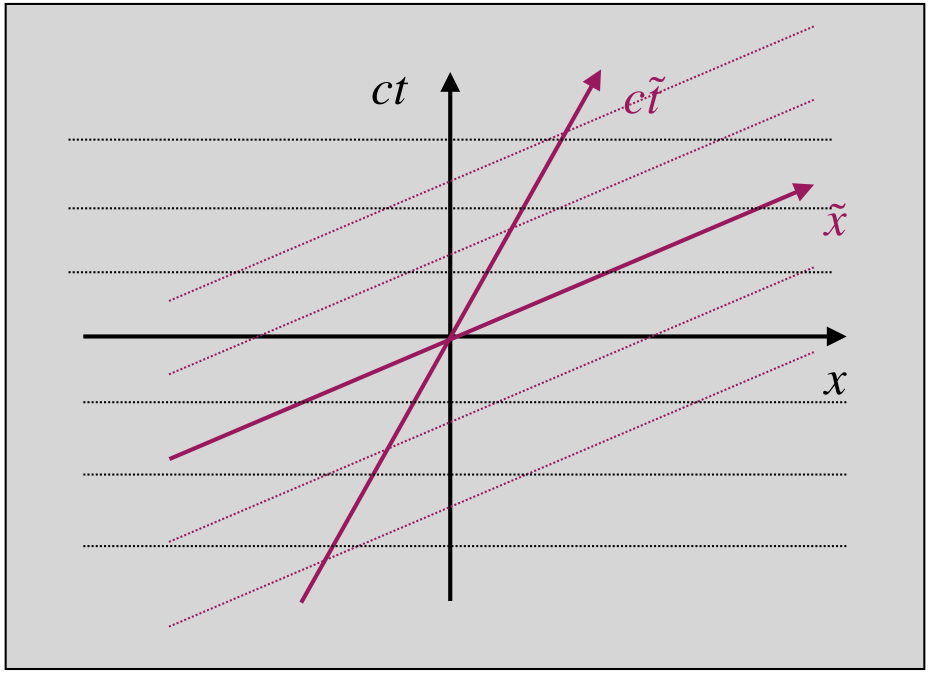
$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\tilde{x} = \gamma\, \frac{v}{c}\, ct - \gamma\, x$$

$$\tilde{t} = t$$

$$\tilde{x} = x - vt$$

Lorentz boosts



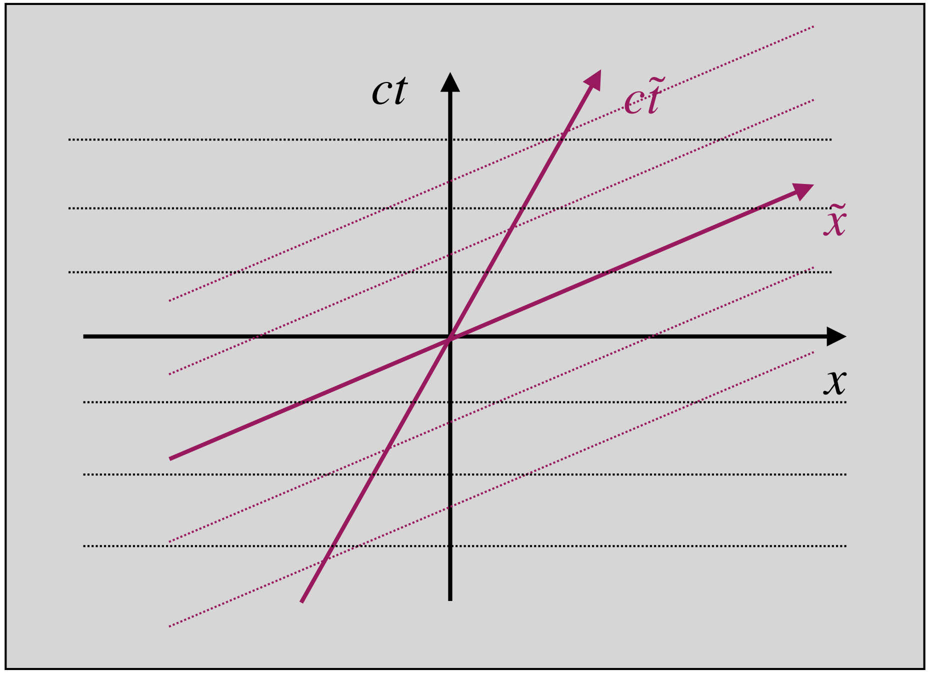
General boost with velocity $\vec{v}$

$$c\tilde{t} = \gamma\, ct - \gamma \sum_{i=1}^{3} \frac{v^i}{c}\, x^i$$

$$\tilde{x}^i = -\gamma\, \frac{v^i}{c}\, ct + \gamma\, x^i_{\parallel} + x^i_{\perp}$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{\vec{v}^{\,2}}{c^2}}}$$

Lorentz boosts



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Inverse relation: $\vec{v} \to -\vec{v}$

$$ct = \gamma\, c\tilde{t} + \gamma \sum_{i=1}^{3} \frac{v^i}{c}\, \tilde{x}^i$$

$$x^i = \gamma \frac{v^i}{c}\, c\tilde{t} + \gamma\, \tilde{x}^i$$

Special relativity

Conventions simplifying life:

measure time with meters $t_{New} = c\, t_{Old}$ [m]

$v_{New} = v_{Old}/c$ [1]

$c_{New} = 1$ simplifies many formulas

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Einstein's summation convention

$$a_\mu b^\mu \equiv \sum_{\mu=0}^{3} a_\mu b^\mu = a_0 b^0 + a_1 b^1 + a_2 b^2 + a_3 b^3$$

Greek indices = 0, 1, 2, 3

$$c_{\mu\nu} d^\mu e^\nu \equiv \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} c_{\mu\nu} d^\mu e^\nu = \sum_{\nu=0}^{3} \sum_{\mu=0}^{3} c_{\mu\nu} d^\mu e^\nu$$

index **contracted** with another index

Special relativity

Conventions simplifying life:

measure time with meters $t_{New} = c\, t_{Old}$ $[\mathrm{m}]$

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$$\begin{aligned} c_{\mu\nu}\, d^\mu\, e^\nu \equiv\ & c_{00}\, d^0\, e^0 + c_{01}\, d^0\, e^1 + c_{02}\, d^0\, e^2 + c_{03}\, d^0\, e^3 \\ & + c_{10}\, d^1\, e^0 + c_{11}\, d^1\, e^1 + c_{12}\, d^1\, e^2 + c_{13}\, d^1\, e^3 \\ & + c_{20}\, d^2\, e^0 + c_{21}\, d^2\, e^1 + c_{22}\, d^2\, e^2 + c_{23}\, d^2\, e^3 \\ & + c_{30}\, d^3\, e^0 + c_{31}\, d^3\, e^1 + c_{32}\, d^3\, e^2 + c_{33}\, d^3\, e^3 \end{aligned}$$

Special relativity

Einstein’s summation condition cont.

$$a_\mu b^\mu = a_\alpha b^\alpha = a_\gamma b^\gamma = \cdots = a_0 b^0 + a_1 b^1 + a_2 b^2 + a_3 b^3$$

$$a_\mu b^\mu = b^\mu a_\mu$$

$$c_{\mu\nu} d^\mu e^\nu = c_{\mu\nu} e^\nu d^\mu = d^\mu e^\nu c_{\mu\nu} = \cdots$$

but $c_{\mu\nu} d^\mu e^\nu \neq c_{\mu\nu} d^\nu e^\mu$

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Dummy (summation) indices vs free indices

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$a_\mu b_\mu, \quad c^\sigma d^\sigma$ undefined

$a_\alpha b^\alpha c_\alpha$ undefined

Special relativity

$(t, x^i) \equiv (x^0, x^i)$

General Lorentz boost to frame moving with velocity $\vec{v}$

$$\begin{pmatrix} \tilde{x}^0 \\ \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\, v^1 & -\gamma\, v^2 & -\gamma\, v^3 \\ -\gamma\, v^1 & 1 + \frac{\gamma - 1}{\vec{v}^2} \left(v^1\right)^2 & \frac{\gamma - 1}{\vec{v}^2} v^1\, v^2 & \frac{\gamma - 1}{\vec{v}^2} v^1\, v^3 \\ -\gamma\, v^2 & \frac{\gamma - 1}{\vec{v}^2} v^1\, v^2 & 1 + \frac{\gamma - 1}{\vec{v}^2} \left(v^2\right)^2 & \frac{\gamma - 1}{\vec{v}^2} v^2\, v^3 \\ -\gamma\, v^3 & \frac{\gamma - 1}{\vec{v}^2} v^1\, v^3 & \frac{\gamma - 1}{\vec{v}^2} v^2\, v^3 & 1 + \frac{\gamma - 1}{\vec{v}^2} \left(v^3\right)^2 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

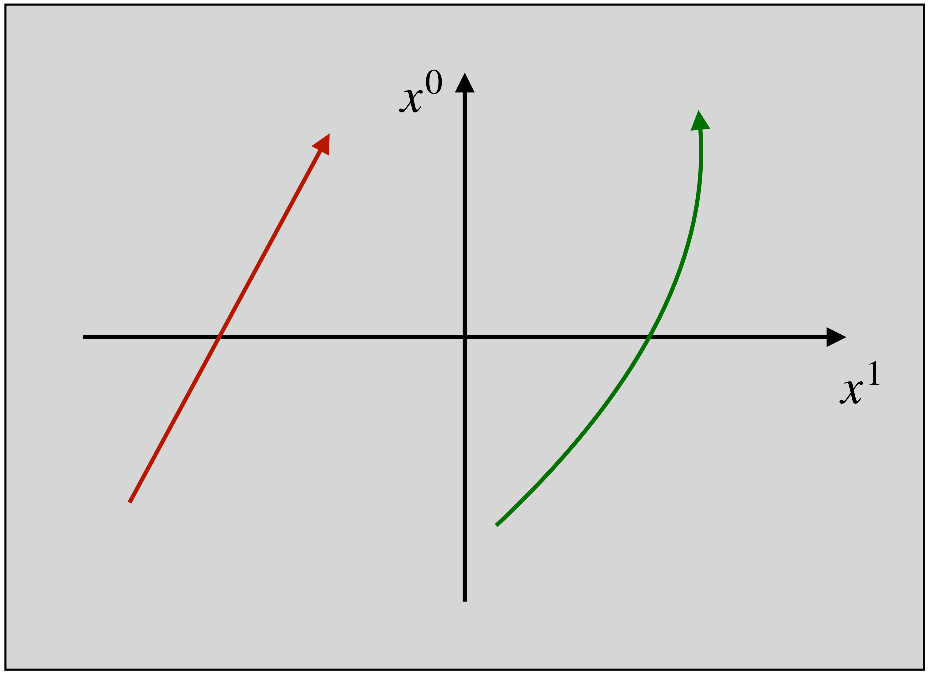
$$\gamma = \frac{1}{\sqrt{1 - \vec{v}^2}}$$

$\Lambda^{\mu}{}_{\nu}(\vec{v})$

$\Lambda(\vec{v})^{-1} = \Lambda(-\vec{v})$

Kinematics in SR

Motions of massive particles described by **worldlines** $x^\mu(\lambda)$



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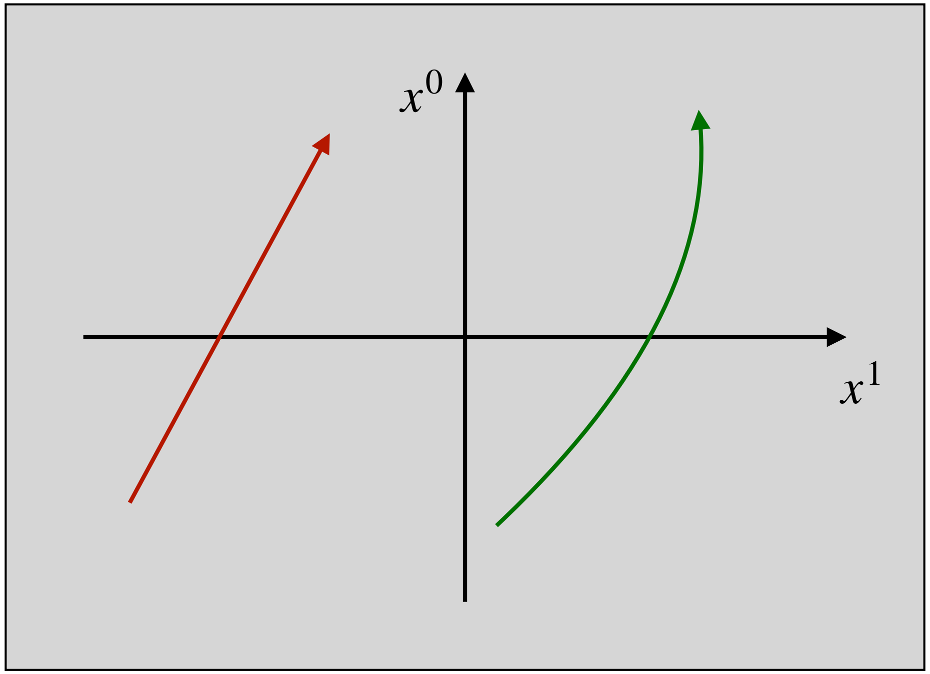

simplest parametrization: **coordinate time**

$$x^\mu(x^0) = \begin{pmatrix} x^0 \\ x^1(x^0) \\ x^2(x^0) \\ x^3(x^0) \end{pmatrix} \qquad \frac{d\,x^\mu(x^0)}{dx^0} = \begin{pmatrix} 1 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix}$$

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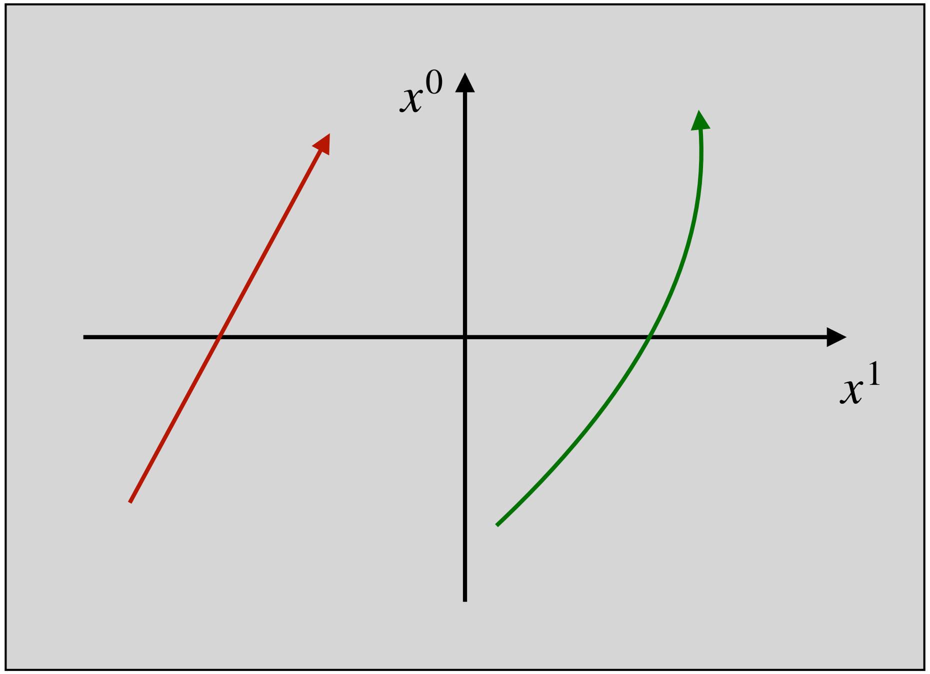

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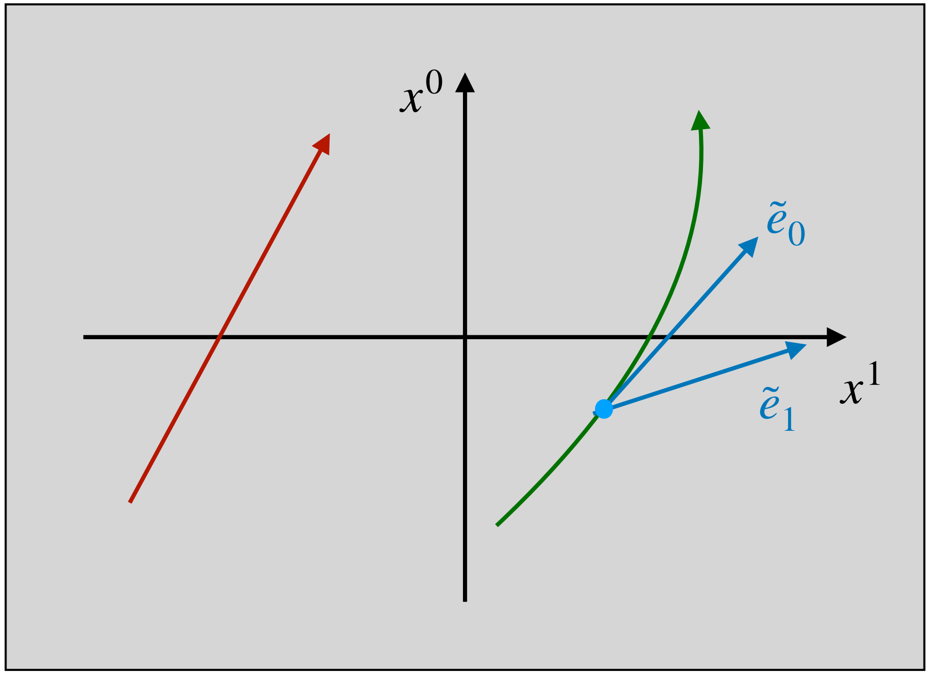

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(momentrarily) **comoving frame**:

$$\tilde{e}^\mu_0 = C \begin{pmatrix} 1 \\ v^i \end{pmatrix}$$

$$\tilde{e}^\mu_0\, \tilde{e}_{0\,\mu} = -1 \quad \Longrightarrow \quad C = \frac{1}{\sqrt{1 - v^2}} \equiv \gamma$$

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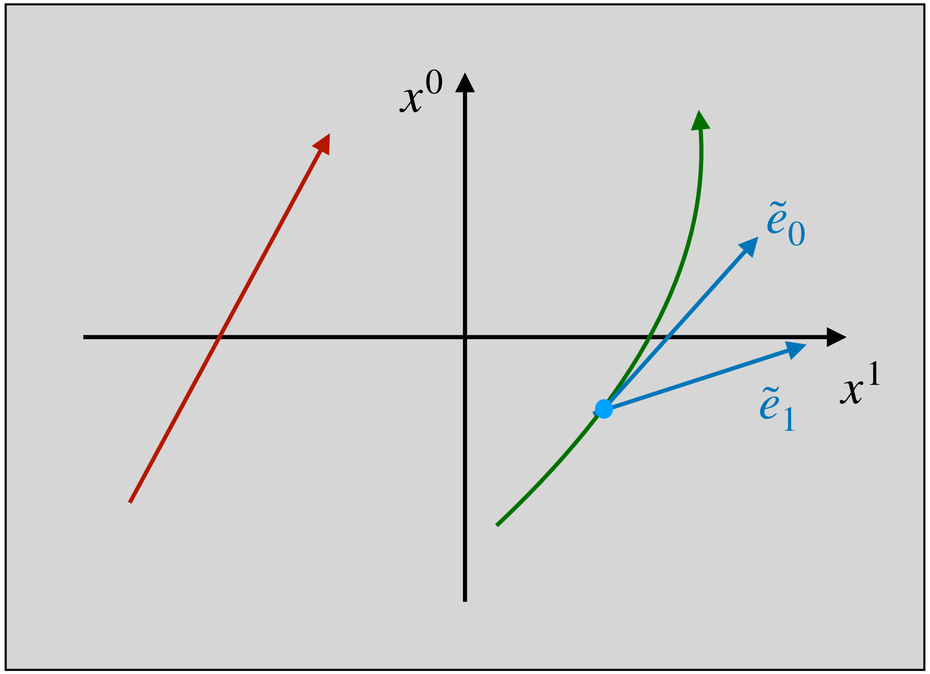

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$$\tilde{e}_0^\mu = u^\mu = \begin{pmatrix} \gamma \\ \gamma v^i \end{pmatrix}$$

Body's momentary **4-velocity** u^μ